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Pure-strategy Nash equilibria in large games: Characterization and Existence[1]

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Abstract

We present three characterization results for pure-strategy Nash equilibria in three different settings of large games, as well as a counterexample showing the failure of this characterization framework in another setting. We then show the existence of the characterizing counterpart for the equilibria, which subsequently enables us to obtain the existence of pure-strategy Nash equilibria in three settings of large games.

Key words: Large games; pure-strategy Nash equilibrium; characterization; existence; atomless probability space; saturated probability space

1. Introduction

A *large game* models its agent space with an atomless probability space, which captures the predominant characteristic in a large conflicting economy where a single player is negligible but a group of players is influential. Over the past few decades, research on large games has been fruitful. Various results on the existence or nonexistence of the pure or mix-strategy Nash

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equilibria are determined in different settings or frameworks of large games.[5]

However, most studies on large games focus on showing the existence or nonexistence of Nash equilibria but very few of them pay attention to characterizing the equilibria, which, in our point of view, is certainly a loss in the literature. As we can see from this paper, not only can a good characterisation result help us explain the equilibria from another perspective and hence enhance our understanding of them, but it can also provide an alternative and even easier way to show the existence of the equilibria.

We start by considering a generalized large game where the agent space is divided into countable (finite or countably infinite) different subgroups and each player's payoff depends on her own action and the action distribution in each of the subgroups.[6] In such a large game, a pure-strategy action profile that assigns an action to each player is called a *(pure-strategy) Nash equilibrium* if no player has the incentive to deviate from her assigned action. A *(pure-strategy) equilibrium distribution* is a distribution on the action space that is induced by a pure-strategy Nash equilibrium.

If such a large games is further restricted by having either *a countable action space* or *a countable payoff space* or *a saturated probability space of agents*, then a given distribution on the action space is shown to be an equilibrium distribution if and only if for any Borel [closed or open (or finite)] subset of actions the players in each subgroup playing actions in it are no more than the players having a best response in the set. We also show through a counterexample that if both actions and payoffs are uncountable and the agent space is a general probability space, say the Lebesgue unit interval, then a similar characterization result is *not* valid.

Following these characterization results, we proceed to show the existence of the characterizing distribution for the equilibrium. Our result (Theorem

[5]Interested readers can refer to Khan and Sun (1999, 2002); Kalai (2004) for a few existing literatures and Khan et al. (2012) for a new development.

[6]The large game discussed here is a generalization to the large non-anonymous games discussed in Khan and Sun (1999, 2002).

5) reveals that the characterizing distributions do exist and they exist in a much more general framework than the equilibria do. In particular, there is no need to impose any further restrictions on the agent, action or payoff space other than the regular conditions that define a large game. This result, together with the previous characterization results, leads to the existence of pure strategy Nash equilibria in three settings of large games under countability or saturation assumption. These existence results generalize or parallel the corresponding results in Khan and Sun (1995, 1999) and also include a new situation showing the existence of pure strategy Nash equilibria in large games endowed with at most countably many different payoffs while dropping any countability or saturation restrictions on the agent or action space.

Throughout the paper, we present quite a few results on the characterization or existence of pure-strategy equilibria in large games. However our paper is not tedious and our proofs are mostly elementary. This can be seen as another advantage of considering the existence of pure strategy equilibrium via its characterization.[7]

The paper is organized as follows. Section 2 introduces the game model. Section 3 presents all the characterization results. Section 4 shows the existence of the characterizing equilibrium and hence also the pure strategy equilibrium. Section 5 contains some concluding remarks.

2. Large game model

Let $(T, \mathscr{T}, \lambda)$ be an atomless probability space of *players* and I a countable (finite or countably infinite) index set. Let $(T_i)_{i\in I}$ be a measurable partition of T with positive λ-measures $(\alpha_i)_{i\in I}$, i.e., $\alpha_i = \lambda(T_i) > 0$ for all

[7]The proof of our first result uses Bollobas and Varopoulos (1975)'s extension of the famous marriage theorem (or the Hall's theorem) and the proof of the third result relies on Keisler and Sun (2009)'s result on the distributional properties of correspondence on saturated probability spaces.

$i \in I$. For each $i \in I$, let $\lambda_i := \lambda|_{T_i}$ be the *restricted probability measure* of λ on T_i in the sense that λ_i is a probability measure on the measurable space $(T_i, \mathscr{T}_i)$ where $\mathscr{T}_i := \{S \cap T_i : S \in \mathscr{T}\}$ such that for any set $B \in \mathscr{T}_i$, $\lambda_i(B) = \lambda(B)/\alpha_i$. By introducing this partition, we imply that the players are divided into I groups.

Let the *action space*, denoted by A, of the game be a Polish space with $\mathscr{B}(A)$ its Borel σ-algebra and $\mathscr{M}(A)$ the set of all Borel probability measures on A. Suppose that all the players in each group $i \in I$ choose their *actions* from a common compact subset A_i of A. Without loss of generality, we assume that $(A_i)_{i\in I}$ are disjoint of each other.[8] For ease of notation, we define an action correspondence $K : T \twoheadrightarrow A$ for all players such that $K(t) = A_i$ for all $t \in T_i$. Let $\mathscr{M}(A_i)$ be the set of all Borel probability measures on A_i endowed with the topology of weak convergence of probability measures and $\prod_{i\in I} \mathscr{M}(A_i)$ the usual product space endowed with the product topology. For ease of notation, we let $\Theta := A \times \prod_{i\in I} \mathscr{M}(A_i)$ and $\Theta_i := A_i \times \prod_{i\in I} \mathscr{M}(A_i)$, $i \in I$. [9]

The *payoff function* (or simply, *payoff*) of each player depends on her own action as well as on the distribution of actions played by the players in each of the groups. Mathematically, we let the space of *payoffs* be the space of all continuous real-valued functions on Θ, denoted by $\mathcal{C}(\Theta)$, endowed with the topology of compact convergence.

Definition 1.
A *large game* is a measurable mapping U from T to $\mathcal{C}(\Theta)$.[10] A measurable function $f : T \to A$ is called a *pure-strategy profile* if $f(t) \in K(t)$ for all

[8]If initially, $(A_i)_{i\in I}$ are not disjoint, we can always introduce a disjoint set of action sets $(A'_i)_{i\in I}$ by adding an index dimension to the original action sets while keeping the same topological structure. For example, if $A_1 = A_2 = \{a, b\}$, we can let $A'_1 = \{(1, a), (1, b)\}$ and $A'_2 = \{(2, a), (2, b)\}$.

[9]Unless otherwise specified, any topological space discussed in this paper is tacitly understood to be equipped with its Borel σ-algebra, i.e., the σ-algebra generated by the family of open sets, and the measurability is defined in terms of it.

[10]Such a large game is often called large non-anonymous game in the literature. See,

$t \in T$. A pure-strategy profile f is called a *pure-strategy (Nash) equilibrium* if for λ-almost all $t \in T$,

$$U(t)[f(t), (\lambda_i f_i^{-1})_{i\in I}] \geq U(t)[a, (\lambda_i f_i^{-1})_{i\in I}] \text{ for all } a \in K(t),$$

where f_i is the restriction of f to T_i. A distribution $\mu \in \mathscr{M}(A)$ is called an *equilibrium distribution* if there exists a pure-strategy equilibrium f such that $\mu = \lambda f^{-1}$.

Given a pure-strategy profile $f : T \to A$ and its induced distribution $\mu := \lambda f^{-1}$, let f_i be the restribution of f to T_i and $\mu_i := \mu|_{A_i}$ the *restricted probability measure* of μ on A_i. Since $(A_i)_{i\in I}$ are disjoint sets, $f_i^{-1}(B) = f^{-1}(B)$ for all $B \in A_i$ and hence for any $B \in \mathscr{B}(A_i)$, $\mu_i(B) = \frac{\mu(B)}{\mu(A_i)} = \frac{\lambda f^{-1}(B)}{\lambda f^{-1}(A_i)} = \frac{\lambda f_i^{-1}(B)}{\lambda f_i^{-1}(A_i)} = \frac{\lambda f_i^{-1}(B)}{\lambda(T_i)} = \lambda_i f_i^{-1}(B)$. Thus we have $\mu_i = \lambda_i f_i^{-1}$ for all $i \in I$.

Recall that a *correspondence* F from T to A, denoted by $F : T \twoheadrightarrow A$, is called *measurable* if for each closed subset C of A, the set

$$F^{-1}(C) := \{t \in T : F(t) \cap C \neq \emptyset\}$$

is measurable in $\mathscr{T}$. A function f from T to A is said to be a *measurable selection* of F if f is measurable and $f(t) \in F(t)$ for all $t \in T$. When F is measurable and closed valued, the classical Kuratowski-Ryll-Nardzewski Theorem (see, eg, Aliprantis and Border (1999, p.567)) shows that F has a measurable selection.

Given an arbitrary probability measure $\mu \in \mathscr{M}(A)$, the *best responses* of player t facing the collective behavior μ is given by

$$B^\mu(t) := \arg\max_{a\in K(t)} U(t)(a, (\mu_i)_{i\in I}),$$

where and μ_i is the restricted probability measure of μ on A_i. By the Measurable Maximum Theorem in Aliprantis and Border (1999, p.570), B^μ is a

eg, Khan and Sun (2002).

measurable correspondence from T to A, has nonempty compact values and admits a measurable selection. Let $B_i^\mu : T_i \twoheadrightarrow A_i$ be the restriction of B^μ to T_i.

3. Characterizing large game

Unless otherwise specified, throughout this section, we follow all the notations defined in last section.

3.1. Large games with countable actions

Our first result is on large games with countable actions.

Theorem 1. *Let $\mu \in \mathscr{M}(A)$ and $\mu_i = \mu|_{A_i}$ for all $i \in I$. If the action space A in the large game U is countable, then the following statements are equivalent:*

(i) *μ is an equilibrium distribution;*

(ii) *for each $i \in I$, $\mu_i(C) \leq \lambda_i[(B_i^\mu)^{-1}(C)]$ for every subset C in A_i;*

(iii) *for each $i \in I$, $\mu_i(D) \leq \lambda_i[(B_i^\mu)^{-1}(D)]$ for every finite subset D in A_i.*

To prove this theorem, we need the following lemma from Khan and Sun (1995), which is a special case of the famous marriage theorem offered by Bollobas and Varopoulos (1975).[11]

Lemma 1. *(Khan and Sun, 1995, Theorem 4)*[12] *Let $(T, \mathcal{T}, \lambda)$ be an atomless probability space, I a countable index set, $(T_i)_{i\in I}$ a family of sets in $\mathcal{T}$, and $(\alpha_i)_{i\in I}$ a family of non-negative numbers. Then the following two statements are equivalent*

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[11] This lemma was also used by Yu and Zhang (2007) to show the existence of pure strategy in games with countable actions.

[12] Throughout the paper, we reference results previously available in the literature as "Lemma".

- *$\lambda(\bigcup_{i\in D} T_i) \geq \sum_{i\in D} \alpha_i$ for all finite subsets D of I;*

- *there is a family of sets, $(S_i)_{i\in I}$, in $\mathcal{T}$ such that for all $i,j \in I, i \neq j$, one has $S_i \subseteq T_i$, $\lambda(S_i) = \alpha_i$ and $S_i \cap S_j = \emptyset$.*

Proof of Theorem 1. (i)⇒(ii): Let μ be an equilibrium distribution. Then by definition, there exists a Nash equilibrium $f : T \to A$ such that $\mu = \lambda f^{-1}$. Notice that for each $i \in I$, $f_i(t) \in B_i^\mu(t)$ for all $t \in T_i$. Thus, for any $i \in I$ and for every $C \subseteq A_i$,

$$\begin{aligned}\mu_i(C) &= \lambda_i(f_i^{-1}(C)) = \lambda_i(\{t \in T_i : f_i(t) \in C\}) \\ &\leq \lambda_i(\{t \in T_i : B_i^\mu(t) \cap C \neq \emptyset\}) = \lambda_i[(B_i^\mu)^{-1}(C)].\end{aligned}$$

(ii) ⇒ (iii): Obvious.

(iii)⇒(i): Suppose (iii) holds. Fix an arbitrary $i \in I$. Since A is a countable set, it can be written as $A := \{a_1, a_2, \cdots\} = \{a_j\}_{j\in\mathbb{N}}$. For each $j \in \mathbb{N}$, let $\beta_j := \mu_i(\{a_j\})$ and $T_i^j := (B_i^\mu)^{-1}(\{a_j\}) = \{t \in T_i : a_j \in B_i^\mu(t)\}$. Let D be an arbitrary finite subset of $\mathbb{N}$. Observe that $(B_i^\mu)^{-1}(\bigcup_{j\in D}\{a_j\}) = \bigcup_{j\in D} T_i^j$. Statement (iii) tells that $\sum_{j\in D}\beta_j = \mu_i(\bigcup_{j\in D}\{a_j\}) \leq \lambda_i(\bigcup_{j\in D} T_i^j)$. Thus by Lemma 1, there exists a family of sets, $(S_j)_{j\in\mathbb{N}}$, such that for all $j,k \in \mathbb{N}, k \neq j$, one has $S_j \subseteq T_i^j$, $\lambda_i(S_j) = \beta_j$ and $S_j \cap S_k = \emptyset$.

Now define a measurable function $h_i : T_i \to A$ such that for all $j \in \mathbb{N}$ and for all $t \in S_j, h_i(t) = a_j$. Since, for any $j \in \mathbb{N}$, $t \in S_j$ implies that $a_j \in (B_i^\mu)(t)$, we have $h_i(t) \in B_i^\mu(t)$ for all $t \in T$. Furthermore, $\lambda_i(h_i^{-1}(\{a_j\})) = \lambda_i(S_j) = \beta_j = \mu_i(\{a_j\})$ for all $j \in \mathbb{N}$, which implies $\lambda_i h_i^{-1} = \mu_i$. Repeat the above arguments for all $i \in I$ and define a measurable function $h : T \to A$ by letting $h(t) = h_i(t)$ if $t \in T_i$. Thus it is clear that h is a pure strategy Nash equilibrium and $\mu = (\mu_i)_{i\in\mathbb{N}} = \lambda h^{-1}$ is the equilibrium distribution induced by h. *Q.E.D.*

Remark 1. Note that μ is an equilibrium distribution if and only if there exists a measurable selection f of B^μ such that $\mu = \lambda f^{-1}$. Hence, if μ is an equilibrium distribution, then $\mu_i(C) = \lambda_i(f_i^{-1}(C)) = \lambda_i\{t \in T_i : f_i(t) \in C\}$, is simply the proportion of players playing their actions in C. Therefore, the above theorem literally says that a distribution on the product action space is an equilibrium distribution if and only if for any subset or any finite subset of the actions, there are less players in each group playing their actions in the subset than having a best response in it. It should be noted that the case that $|I| = 1$ and A is finite in our theorem is the main result in Blonski (2005).

3.2. large games with countable payoffs

In the last section, we characterize large games with a countable set of actions. One may wander if we can allow an action space without the countability restriction. The answer is yes provided that there are only countably many payoff functions in the game, or equivalently, all the players in each group play a common payoff function.

Definition 2. The players in a large game U is said to be *homogeneous* if for each group $i \in I$, $U(t)$ is same for all $t \in T_i$.

Since the total number of elements in a countable collection of countable sets is still countable, this definition of homogeneity is equivalent to assuming that in each group there are at most countably many payoff functions for its players.

Theorem 2. *Let $\mu \in \mathscr{M}(A)$ and $\mu_i = \mu|_{A_i}$ for all $i \in I$. If the players in the large game U is homogeneous, then the following statements are equivalent:*

(i) *μ is an equilibrium distribution;*

(ii) *for each $i \in I$, $\mu_i(C) \leq \lambda_i[(B_i^\mu)^{-1}(C)]$ for every Borel subset C in A_i;*

(iii) *for each $i \in I$, $\mu_i(D) \leq \lambda_i[(B_i^\mu)^{-1}(D)]$ for every closed subset D in A_i;*

(iv) *for each $i \in I$, $\mu_i(O) \leq \lambda_i[(B_i^\mu)^{-1}(O)]$ for every open subset O in A_i.*

To prove this theorem, we firstly introduce the following well known lemma which can be obtained by appropriately adjusting the proof of Theorem 3.11 in Skorokhod (1956).

Lemma 2. *(Skorokhod, 1956, Theorem 3.11) Let $(T, \mathcal{T}, \lambda)$ be an atomless probability space and A a Polish space. Then for any $\nu \in \mathscr{M}(A)$ there exists a measurable function $f : T \to A$ such that $\lambda f^{-1} = \nu$.*

Proof of Theorem 2. Firstly, we want to make sure that for each $i \in I$ and every $C \in \mathscr{B}(A_i)$, $(B_i^\mu)^{-1}(C)$ is measurable. To see this, fix any $i \in I$. The homogeneous condition, i.e., $U(t)$ is fixed for all $t \in T_i$, implies that $B_i^\mu(t)$ is same for all $t \in T_i$. Thus we can let $C_i := B_i^\mu(t)$ for all $t \in T_i$. Then, for any $C \in \mathscr{B}(A_i)$, we have

$$(B_i^\mu)^{-1}(C) = \{t \in T_i : B_i^\mu(t) \cap C \neq \emptyset\} = \begin{cases} T_i & \text{if } C_i \cap C \neq \emptyset; \\ \emptyset & \text{otherwise,} \end{cases}$$

which is measurable.

(i)⇒(ii): Suppose μ is now an equilibrium distribution. By assumption, there exists a Nash equilibrium $f : T \to A$ such that $\mu = (\lambda_i f_i^{-1})_{i \in I}$ and $f(t) \in B^\mu(t)$ for all $t \in T$. Therefore, for any $C \in \mathscr{B}(A_i)$,

$$\begin{aligned} \mu_i(C) &= (\lambda_i f_i^{-1})(C) = \lambda_i(\{t \in T_i : f_i(t) \in C\}) \\ &\leq \lambda_i(\{t \in T_i : B_i^\mu(t) \cap C \neq \emptyset\}) \\ &= \lambda_i[(B_i^\mu)^{-1}(C)]. \end{aligned}$$

It is clear that (ii) ⇒ (iii).

(iii) ⇒ (iv): Let O be an open set in A_i. Then there is an increasing sequence $\{F_n\}_{n=1}^\infty$ of closed sets in A_i such that $O = \bigcup_{n=1}^{\infty} F_n$. For each n, we have $(B_i^\mu)^{-1}(F_n) \subseteq (B_i^\mu)^{-1}(O)$, which implies that $\mu_i(F_n) \leq \lambda_i[(B_i^\mu)^{-1}(F_n)] \leq \lambda_i[(B_i^\mu)^{-1}(O)]$. Thus, $\mu_i(O) \leq \lambda_i[(B_i^\mu)^{-1}(O)]$.

It remains to show (iv) ⇒ (i).

Recall that for all $i \in I$, the set $C_i := B_i^\mu(t)$ for any $t \in T_i$ is compact and hence also complete and separable. Fix any $i \in \mathbb{N}$. By the fact that the set $(A_i - C_i)$ is open, we have that

$$1 - \mu_i(C_i) = \mu_i(A_i - C_i) \leq \lambda_i[(B_i^\mu)^{-1}(A_i - C_i)] = 0, \tag{1}$$

which gives $\mu_i(C_i) = 1$ for all i. Therefore, by Lemma 2, there exists a measurable function $f_i : T_i \to C_i$ such that $\mu_i = \lambda_i f_i{}^{-1}$. By definition, $f_i \in B_i^\mu$.

Define $f : T \to A$ by letting $f(t) = f_i(t)$ for all $t \in T_i$ and all $i \in I$. Thus f is a measurable selection of B^μ and $\mu = (\mu_i)_{i\in I} = (\lambda_i f_i^{-1})_{i\in I}$ is an equilibrium distribution.

Q.E.D.

3.3. Large games without countability restrictions

Now one may ask: is there a similar characterization result for a large game in the plain form defined in Section 2? Our next result says no to this question by showing that our characterization result doesn't work for a large game endowed with uncountable actions, uncountable payoffs and a Lebesgue measure space of agents.

Theorem 3. *Let $\mu \in \mathscr{M}(A)$ and $\mu_i = \mu|_{A_i}$ for all $i \in I$. There exists a large game U such that the following statements are* not *equivalent:*

(i) *μ is an equilibrium distribution of U;*

(ii) *for each $i \in I$, $\mu_i(C) \leq \lambda_i[(B_i^\mu)^{-1}(C)]$ for every Borel subset C in A_i;*

To show this result, we need only to give one countable example.

Example 1. Consider a large game U given as follows. Let the space of players be the Lebesgue unit interval $T = [0, 1]$ endowed with its Boral σ-algrbra and the Lebesgue measure λ. Let the action space A be the interval $[-1, 1]$ and the payoffs given by $U(t)(a, \mu) = -|t - |a||$[13] where $t \in T$, $a \in A$ and $\mu \in \mathscr{M}(A)$.

Let η be the uniform distribution on $[-1, 1]$. Thus, given η, the best response set for player t is:

$$B^\eta(t) = \arg \max_{a\in[-1,1]} U(t)(a, \eta) = \{t, -t\}.$$

[13]This payoff function is similar to a payoff function used in Khan et al. (1997).

Let C be an arbitrary Borel subset in A and let $C_1 = C \cap (0,1]$ and $C_2 = C \cap [-1,0]$. Then

$$\begin{aligned}\lambda[(B^\eta)^{-1}(C)] &= \lambda(\{t \in T : B^\eta(t) \cap C \neq \emptyset\}) \\ &= \lambda\{t \in T : t \in C_1 \text{ or } -t \in C_2\} \\ &\geq \max\{\lambda(C_1), \lambda(C_2)\} \\ &\geq \frac{\lambda(C_1) + \lambda(C_2)}{2}.\end{aligned}$$

Since η is the uniform distribution on $[-1,1]$, $\eta(C) = \frac{\lambda(C)}{2} = \frac{\lambda(C_1 \bigcup C_2)}{2} = \frac{\lambda(C_1)+\lambda(C_2)}{2}$. Therefore, we have

$$\lambda[(B^\eta)^{-1}(C)] \geq \eta(C).$$

Now we shall prove by contradiction that η can not be an equilibrium distribution.

Suppose η is an equilibrium distribution. Then, by definition, there exists a measurable selection f of B^η such that $\lambda f^{-1} = \eta$ and $f(t) \in B^\eta(t)$ for all $t \in T$. Let $D = f^{-1}((0,1])$. Then

$$f(t) = \begin{cases} t, & t \in D \\ -t, & t \notin D. \end{cases}$$

Note that $f^{-1}(D) = \{t : f(t) \in D\} = \{t : t \in D\} = D$. Hence, $\lambda(D) = \lambda f^{-1}(D) = \eta(D) = \frac{\lambda(D)}{2}$, which is a contradiction. Therefore, η cannot be an equilibrium distribution. ■

3.4. Large games with agent space being a saturated probability space

Although a general characterization result for equilibria in large games in its plain setting fails to hold as we have seen from last section, we notice that if we assume the agent space to be a saturated probability space, then we can still have a similar characterization result. This result follows easily from the work of Sun (1996) and Keisler and Sun (2009).

To introduce the concept of a saturated probability space, we first recall that a probability space is said to be *countably-generated* if its σ-algebra can be generated by countably many sets; otherwise it is *not countably-generated*.

Definition 3. A probability space $(T, \mathscr{T}, \lambda)$ is said to be *saturated* if it is nowhere countably-generated, in the sense that, for any subset $C \in T$ with $\lambda(C) > 0$, the restricted probability space $(C, \mathscr{T}_C, \lambda_C)$ is not countably-generated, where $\mathscr{T}_C := \{C \cap C' : C' \in \mathscr{T}\}$ and λ_C is the probability measure derived from the restriction of λ to $\mathscr{T}_C$.

Or equivalently,

Definition 4. A probability space $(T, \mathscr{T}, \lambda)$ is *saturated* if it is atomless and for every Borel probability measure ν on the product of Polish spaces $X \times Y$ if for every measurable mapping $f : T \to X$ which induces the distribution as the marginal measure of ν over X, then there is a measurable mapping $g : T \to Y$ such that the induced distribution of the pair (f, g) on $(T, \mathscr{T}, \lambda)$ is ν.[14]

Note that in our Example 1, the Lebesgue unit interval $[0, 1]$ endowed with its σ-algebra of Lebesgue measurable sets and the Lebesgue measure, is a countably-generated probability space, and hence *not* saturated.

Theorem 4. *If the agent space $(T, \mathscr{T}, \lambda)$ of the large game U is a saturated probability space, then Theorem 2 still holds if we discard the homogeneous assumption.*

To prove the above theorem, we shall refer to the following lemma which is analogous to Proposition 3.5 of Sun (1996)

Lemma 3. *Let F be a closed valued measurable correspondence from a saturated probability space $(\Omega, \mathscr{F}, P)$ to a Polish space X. Let ν be a Borel probability measure on X. Then the following statements are equivalent:*

(i) *there is a measurable selection f of F such that $Pf^{-1} = \nu$;*

(ii) *for every Borel set C in X, $\nu(C) \leq P(F^{-1}(C))$;*

[14] See Keisler and Sun (2009) or Khan et al. (2012) for a discussion on the equivalence of the two definitions.

(iii) *for every closed set D in X, $\nu(D) \leq P(F^{-1}(D))$;*

(iv) *for every open set O in X, $\nu(O) \leq P(F^{-1}(O))$.*

Proof. This lemma is analogous to Proposition 3.5 of Sun (1996). It follows easily by combining (Keisler and Sun, 2009, P3, Theorem 3.6) and (Keisler and Sun, 2009, Proposition 3.5). *Q.E.D.*

Proof of Theorem 4. For any $i \in I$, notice that B_i^μ is a compact valued (and hence closed valued) measurable correspondence from an atomless Loeb probability space $(T_i, \mathscr{T}_i, \lambda_i)$ to the Polish space A. Thus, by applying Lemma 3 to B_i^μ, we see that $\mu_i = \lambda_i f_i^{-1}$ for some f_i being a measurable selection of B_i^μ if and only if for every Borel (closed, or open) set H in A_i, $\mu_i(H) \leq \lambda_i[(B_i^\mu)^{-1}(H)]$.

Since the above result holds for all $i \in I$, thus $\mu = (\mu_i)_{i\in I}$ is an equilibrium distribution if and only if for each $i \in I$ and every Borel (closed, or open) set H in A_i, $\mu_i(H) \leq \lambda_i[(B_i^\mu)^{-1}(H)]$.

Q.E.D.

4. Existence of equilibrium in large game

The above characterization results enables to understand the equilibria in large games from another perspective. Moreover these characterization results also pave us for another way to show the existence of equilibrium distributions and hence pure-strategy profiles.

Theorem 5. *There exists in every large game U a distribution $\mu \in \mathscr{M}(A)$ such that for each $i \in I$,*

$$\mu_i(E) \leq \lambda_i[(B_i^\mu)^{-1}(E)] \text{ for every Borel set } E \text{ in } A_i.$$

where μ_i is the restricted probability measure of μ on A_i.

Proof of Theorem 5. Let $\mu_i \in M(A_i)$ for all $i \in I$. For easy notation, we use $\bar{\mu}$ to denote the distribution vector $(\mu_1, \mu_2, \cdots)$, i.e., $\bar{\mu} := (\mu_i)_{i\in I}$. Also define $B^{\bar{\mu}}(t) := \arg\max_{a\in K(t)} U(t)(a, (\mu_i)_{i\in I})$, which is the best response correspondence.

Now for each group $i \in I$, let $B_i^{\bar{\mu}} : T_i \twoheadrightarrow A_i$ be the restriction of $B^{\bar{\mu}}$ to T_i and $U_i : T_i \to \mathcal{C}(\Theta)$ the restriction of U to T_i. Define $V_i : T_i \to \mathcal{C}(\Theta_i)$ by letting $V_i(t) = U_i(t)|_{\Theta_i}$, where $U_i(t)|_{\Theta_i}$ is the restriction of $U_i(t)$ to Θ_i and $\mathcal{C}(\Theta_i)$ is also endowed with the topology of compact convergence. Thus, we also have $B_i^{\bar{\mu}}(t) = \arg\max_{a\in A_i} V_i(t)(a, (\mu_i)_{i\in I})$. As mentioned earlier in the paper, each topological space is endowed with its Borel σ-algebra on which we define the measurability.

Now we claim that V_i is also measurable. To see this, we first define $W_i : \mathcal{C}(\Theta) \to \mathcal{C}(\Theta_i)$ by letting $W_i(u) = u|_{\Theta_i}$ for all $u \in \mathcal{C}(\Theta)$. Thus $V_i = W_i \circ U_i$ and hence we only need to show that W_i is measurable. Let d be the usual metric on $\mathbb{R}$. Given an element f of $\mathcal{C}(\Theta_i)$, a compact subset D of Θ_i and a number $\epsilon > 0$, let $B_{\Theta_i}(f, D, \epsilon) = \{g \in \mathcal{C}(\Theta_i) : \sup\{d(f(x), g(x))|x \in D\} < \epsilon\}$. Thus the sets $B_{\Theta_i}(f, D, \epsilon)$ form a basis for the topology of compact convergence on $\mathcal{C}(\Theta_i)$.(See, eg, p 283 in Munkres (2000)) Hence we only need to show that $W_i^{-1}(B_{\Theta_i}(f, D, \epsilon))$ is measurable. To see this, let $\Delta = \{u \in \mathcal{C}(\Theta) : u|_D = f\}$ and note that

$$\begin{aligned}
W_i^{-1}(B_{\Theta_i}(f, D, \epsilon)) &= \{h \in \mathcal{C}(\Theta) : h|_{\Theta_i} \in B_{\Theta_i}(f, D, \epsilon)\} \\
&= \{h \in \mathcal{C}(\Theta) : \sup\{d(f(x), h(x))|x \in D\} < \epsilon\} \\
&= \bigcup_{u\in\Delta} \{h \in \mathcal{C}(\Theta) : \sup\{d(u(x), h(x))|x \in D\} < \epsilon\} \\
&= \bigcup_{u\in\Delta} B_{\Theta}(u, D, \epsilon).
\end{aligned}$$

Since $B_\Theta(u, D, \epsilon)$ is open by the definition of the topology on $\mathcal{C}(\Theta)$, $W_i^{-1}(B_{\Theta_i}(f, D, \epsilon))$ is also open and hence measurable. Thus our claim is verified.

For all $i \in I$, define $\Gamma_i^{\bar{\mu}} : \mathcal{C}(\Theta_i) \twoheadrightarrow A_i$ by letting $\Gamma_i^{\bar{\mu}}(u) = \arg\max_{a\in A_i} u(a, (\mu_i)_{i\in I})$ for all $u \in \mathcal{C}(\Theta_i)$. Thus we have $B_i^{\bar{\mu}}(t) = \Gamma_i^{\bar{\mu}}(V_i(t))$ for all $t \in T_i$. By the

Berge's Maximum Theorem, $\Gamma_i^{\bar{\mu}}$ is upper semicontinuous.[15] Thus, $(\Gamma_i^{\bar{\mu}})^{-1}(F)$ is measurable for all closed set $F \in A$.[16] It is also straightforward to verify that $V_i^{-1}[(\Gamma_i^{\bar{\mu}})^{-1}(F)] = (B_i^{\bar{\mu}})^{-1}(F)$ for any closed set $F \in A$. Since V_i is measurable, $\lambda_i V_i^{-1}$ is a Borel probability measure on $\mathcal{C}(\Theta_i)$.

Let $\bar{\eta} := (\eta_i)_{i\in I} \in \prod_{i\in I} \mathscr{M}(A_i)$. Define $\Phi : \prod_{i\in I} \mathscr{M}(A_i) \twoheadrightarrow \prod_{i\in I} \mathscr{M}(A_i)$ as

$$\Phi(\bar{\mu}) = \{\bar{\eta} : \eta_i(E) \le \lambda_i[(B_i^{\bar{\mu}})^{-1}(E)] \text{ for each } i \in I \text{ and any } E \in \mathscr{B}(A_i)\}.$$

It is easy to see that Φ is nonempty,[17] closed-valued and convex-valued.

Now we want to show that Φ is upper semicontinuous or, equivalently, has a closed graph. Toward this end, we choose a sequence $\{(\bar{\mu}^m, \bar{\eta}^m)\}_{m\in\mathbb{N}}$ from $\left(\prod_{i\in I} \mathscr{M}(A_i) \times \prod_{i\in I} \mathscr{M}(A_i)\right)$ with $\bar{\eta}^m \in \Phi(\bar{\mu}^m)$ for each m and converging to $(\bar{\mu}^0, \bar{\eta}^0)$. We need to show that $\bar{\eta}^0 \in \Phi(\bar{\mu}^0)$.

Fix any $i \in I$. Let F be a closed subset of A_i and let $\Lambda_m := (\Gamma_i^{\bar{\mu}^m})^{-1}(F)$ and $\Lambda_0 := (\Gamma_i^{\bar{\mu}^0})^{-1}(F)$. Since $\Gamma_i^{\bar{\mu}^0}$ is upper semicontinuous and F is closed, Λ_0 is also closed. Since Θ_i is compact, $\mathcal{C}(\Theta_i)$ is metrizable and we let $\hat{d}$ be one of the compatible metrics on $\mathcal{C}(\Theta_i)$. For all $k = 1, 2, \ldots$, let $G_k = \{u \in \mathcal{C}(\Theta_i) : \hat{d}(u, \Lambda_0)\} < \frac{1}{k}\}$.

Fix any k. We claim that $\Lambda_m \subset G_k$ for large enough m. To see this, let $u_m \in \Lambda_m$, which, by the definition of Λ_m, implies that there is an $a_m \in F$ such that $u_m(a_m, \bar{\mu}^m) = \max_{a\in A_i} u_m(a, \bar{\mu}^m)$. Since $\bar{\mu}^m \to \bar{\mu}^0$ and u_m is uniformly continuous on $A_i \times \prod_{i\in I} \mathscr{M}(A_i)$[18], when m is large enough we have $|u_m(a_m, \bar{\mu}^0) - \max_{a\in A_i} u_m(a, \bar{\mu}^0)| < \frac{1}{k}$. Thus it is straightforward to find a continuous real function $u'_m \in \mathcal{C}(\Theta_i)$ such that $u'_m(a_m, \bar{\mu}^0) = \max_{a\in A_i} u'_m(a, \bar{\mu}^0) = \max_{a\in A_i} u_m(a, \bar{\mu}^0)$ and $\hat{d}(u_m, u'_m) < \frac{1}{k}$.[19] Thus $u'_m \in \Lambda_0$

[15] Note that the map $f_{\bar{\mu}} : A \times \mathscr{U} \to R$ defined by $f_{\bar{\mu}}(a, u) = u(a, (\bar{\mu}_i)_{i\in I})$ is continuous (see Theorem 46.10 in Munkres (2000)).

[16] See, eg, Lemma 16.4 in Aliprantis and Border (1999).

[17] By the Measurable Maximum Theorem, $B_i^{\bar{\mu}}$ admits a measurable selection g_i. Thus $\bar{\eta} = (\lambda_i g_i^{-1})_{i\in I}$ is a trivial element of $\Phi(\bar{\mu})$.

[18] Continuous real function on compact metric space is also uniformly continuous.

[19] Just let u'_m be a little bit bigger than u_m around the area of a_m.

and $u_m \in G_k$.

Hence, the above result and our hypothesis imply that $\bar{\eta}_i^m(F) \leq \lambda_i V_i^{-1}(\Lambda_m) \leq \lambda_i V_i^{-1}(G_k)$ for large enough m. Since $\bar{\eta}_i^m(F) \to \bar{\eta}_i^0(F)$, we have that $\bar{\eta}_i^0(F) \leq \lambda_i V_i^{-1}(G_k)$. Since $G_k \downarrow \Lambda_0$, we have $\bar{\eta}_i^0(F) \leq \lambda_i V_i^{-1}(\Lambda_0) = \lambda_i V_i^{-1}[(\Gamma_i^{\bar{\mu}^0})^{-1}(F)] = \lambda_i[(B_i^{\bar{\mu}^0})^{-1}(F)]$.

Now we want to show the above result holds for all Borel set $E \in A$. To see this, recall that every probability measure on a Polish space is regular.[20] Therefore, we have

$$\begin{aligned} \bar{\eta}_i^0(E) &= \bar{\eta}_i^0(E \cap A_i) = \sup\{\bar{\eta}_i^0(F) : F \text{ is closed and } F \subseteq E \cap A_i\} \\ &\leq \sup\{\lambda_i[(B_i^{\bar{\mu}^0})^{-1}(F)] : F \text{ is closed and } F \subseteq E \cap A_i\} \\ &\leq \lambda_i[(B_i^{\bar{\mu}^0})^{-1}(E \cap A_i)] = \lambda_i[(B_i^{\bar{\mu}^0})^{-1}(E)]. \end{aligned}$$

Since the above arguments hold for all $i \in I$, we conclude that $\bar{\eta}^0 \in \Phi(\bar{\mu}^0)$. Therefore Φ also has a closed graph, hence by the Ky Fan fixed point theorem in Fan (1952), there is a fixed point $\bar{\mu}^* \in \Phi(\bar{\mu}^*)$.

Define a probability measure μ such that $\mu|_{A_i} = \mu_i^*$ for all $i \in I$ and 0 otherwise. Then μ the probability measure that we seek. *Q.E.D.*

Remark 2. Theorem 5 does not impose any restrictions on the agent, payoff and/or action spaces and hence is a quite general result.

Combining Theorems 1, 2, 3, and 5 leads us to the existence of pure-strategy equilibria in large games.

Theorem 6. *If a large game U also satisfies one of the following three conditions:*

(a). the action space A is a countable set;
(b). all the players in each group share a common payoff;
(c). the agent space $(T, \mathscr{T}, \lambda)$ is a saturated probability space,

[20] See Theorem 10.7 in Aliprantis and Border (1999).

then there exists a pure-strategy equilibrium for the game.

Remark 3. By allowing $|I|$ to be countably infinite and the action space A to be Polish, our case (a) is a generalization to Theorem 10 in Khan and Sun (1995) and Theorem 3.2 in Yu and Zhang (2007) and our case (c) strengthens Theorem 1 in Khan and Sun (1999). Moreover, our case (b) is new.

Remark 4. The existence results in Theorem 6 are obtained easily. However, this is not the case if we want to prove those results directly. The direct proofs on the existence of equilibria for the three settings of large games need to be constructed individually and each of them may well involve a lot of effort. (see, eg, Khan and Sun (1995, 1999).)

5. Concluding remarks

We present the three characterization results in this paper which are easy to understand. The counterexample shows that our characterization results are actually quite sharp. These results are also served as a practical tool to determine the pure-strategy Nash equilibria by showing the existence of their characterizing counterparts. We also hope our method can provide some insight to other similar circumstances in game theory or other field.

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Arbitrage Bounds on Currency Basket Options

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Abstract

This paper exploits arbitrage valuation bounds on currency basket options. Instead of using a sophisticated model to price these options, we consider a set of pricing models that are consistent with the prices of available hedging assets. In the absence of arbitrage, we find a pair of valuation bounds on currency basket options that represent a price range for barrier options. Our results extend the work of Hobson, Laurence and Wang (2005a, 2005b) by seeking tight arbitrage valuation bounds. These bounds are enforced by static portfolios that consist of relevant options on individual currency pairs.

Key words: Currency Basket Options; Static Hedging; Arbitrage Bounds

1 Introduction

For many corporations and financial institutions, basket options are an important tool in managing currency exposure. In this article, we derive new results relating the prices of currency basket options to the prices of standard currency option contracts. The need for basket options arises naturally in practice. For example, a company that purchases products from a variety of countries may be exposed to a change in the value of a basket of currencies against its home currency. In seeking to manage its foreign currency exposure, the company could use relevant options on each foreign currency separately. But this way would be inefficient when the changes in one currency is offset by a change in other currency to which the company is exposed. Basket options whose payoffs are based on multiple currency pairs are traded as alternative instruments to manage currency exposure.

A number of pricing models have been proposed to price and hedge basket options after careful calibration to market prices of options on individual underlying assets.[1] On the one hand, a wide spectrum of parametric models are available to practitioners. Bates (1996) presents a stochastic volatility jump-diffusion model to explain the skewness and excess kurtosis implicit in currency option prices. Bollen, Gray and Whaley (2000) suggest a regime-switching model and document that the market prices of currency options

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[1] Although the analytical formula for basket options is unattainable, there exist a number of numerical techniques for pricing and hedging them, for instance, Ashraff, Tarczon and Wu (1995) for a variance-minimizing hedge; Ju (2002) for the method of characterization functions; Brigo, Mercurio, Rapisarda and Scotti (2004) for the moment-matching approach; Pellizzari (2005) for Monte-Carlo simulation.

do incorporate some regime-switching information. Daal and Madan (2005) propose a pure jump model, termed as the variance-gamma (VG) model, to capture large movements in exchange rates. Recently, Carr and Wu (2007) and Bakshi, Carr and Wu (2008) develop stochastic skew models to generate both stochastic volatility and stochastic skewness which are documented in currency options. On the other hand, some researchers are interested in copula theory. A copula function is used to construct a multivariate density distribution in order to be consistent with the market prices of traded assets. Both Cherubini and Luciano (2002) and Rosenberg (2003) propose approaches to price basket options with two underlying assets through copula functions.

However, these models are easily mis-specified, because of little information about which is the correct model. A pricing model delivers the precise and fair price for a basket option, only if this model is the true representation of reality. As a result, the way of model building in turn introduces an uncertainty in the choice of model. In this paper, we tackle the problem of pricing currency basket options from a different perspective. Rather than using a single parametric model, we consider a set of pricing models that are consistent with the observed prices of traded assets. The aim is to derive model-independent price bounds on currency basket options. These bounds are robust to model misspecification in the sense that they gives no-arbitrage bounds which are little dependent on the choice of specific pricing models. More importantly, they are enforced by static replicating portfolios that are constructed from available hedging instruments at inception when an investor enters into a trading position in a currency basket option.

A set of currency options is identified as hedging instruments. These assets provide a wide range of hedging strategies for investors. The underlying argument is that vanilla options determine the marginal risk adjusted probability density of exchange rate prices, but they do not determine either the complete terminal density or the dynamics of exchange rates. Fitting a model to the prices of vanilla options and using the model for designing a dynamic hedge are subject to errors. Also, it is difficult to perfectly hedge a basket option using option portfolios. These concerns make super-replicating strategies useful. These trading strategies have the appealing features of model independence and simplicity, and they only require static positions in hedging instruments at inception. [2]

Lamberton and Lapeyre (1992) first suggest that basket options could be hedged using portfolios of underlying assets. Bertsimas and Popescu (2002) investigate the super-replication of financial derivatives, including basket options and other exotics. Given the knowledge about the moments of return distributions or the prices of relevant hedging assets, they propose a convex optimization method to derive valuation bounds on complex financial derivatives. This method is closely related to the techniques developed by Gotoh and Konno (2002) who propose an efficient algorithm to deal with semidefinite programmes in order to attain valuation bounds on basket options.

D'Aspremont and EI-Ghaoui (2003) and Pena, Vera and Zuluaga (2006) apply the linear programming (LP) approach to price basket options and suggest that valuation bounds on these options can be derived from the prices of other relevant basket options. Laurence and Wang (2005) investigate the relation between pricing and hedging basket options. In these three papers, valuation bounds on basket options with two underlying assets may be expressed analytically. In particular, Laurence and Wang (2005) assume

[2]The way of super-replication in incomplete financial markets can be linked back to the early works of Kramkov (1994) and EI Karoui and Quenez (1995).

that there is only one strike for each individual option.

Hobson, Laurence and Wang (2005a, 2005b) extend Laurence and Wang's results when traded options are available at a continuum of strike prices. In the first paper, they use a Lagrange optimization approach to characterize optimal strikes. A super-replicating strategy that enforces an upper bound is simply a linear combination of European call options. To support upper price bounds, underlying asset processes must be comonotonic. In the second paper, they construct the so-called "STP" portfolios to sub-replicate basket options. Countermonotonic underlying processes yield lower price bounds. Most recently, Chen, Deelstra, Dhaene and Vanmaele (2008) construct static super-replicating strategies for a class of exotic options written on a weighted sum of asset prices, including Asian options and basket options among others. Based on the theory of integral stochastic orders, they provide a characterization for optimal strikes which is different from the methodology proposed by Hobson, Laurence and Wang (2005a).

This paper is motivated by an attempt to extend the work of Hobson, Laurence and Wang (2005a, 2005b). They have derived arbitrage bounds on basket option using portfolios of options on individual underlying assets. In terms of pricing basket options on currencies, we make use of the fact that there typically exist deep and liquid markets in all currency pairs, and the prices of cross-rate options are actually attainable (by contrast there is little trading in options to trade one share for another). These option prices carry useful information about the joint distribution of underlying currencies, and thus valuation bounds on currency basket options could be tightened. These valuation bounds are enforced by static portfolios of both cross-rate options and options denominated in the numeraire currency.

This paper is organized as follows. Section 2 introduces the setup. The properties of both the marginal and joint densities of underlying currency pairs are discussed. Section 3 presents main results. Section 4 provides a numerical analysis regarding both dominating (dominated) strategies and joint distributions. The final section concludes.

2 Preliminaries

Consider a single-period setting in a frictionless currency market (i.e., no short sale restrictions, transaction costs and other frictions). Within this setup, all investments are made at time zero, and all payments are received at time T. There are three main currencies, the Euro (EUR, €), British pound (GBP, £) and U.S. dollar (USD, \$). The interest rates in all currencies are zero. Let the dollar (\$) be the numeraire currency. The positive variables X and Y represent the (unknown) dollar-denominated prices of the Euro and British pound at maturity,

$$X, Y \in \mathbb{R}^{+}. \tag{1}$$

Their time-0 prices are $\$x_0$ (> 0) and $\$y_0$ (> 0). We hereafter also use them to indicate the corresponding foreign currencies unless otherwise specified. Let the variable $Z = Y/X$ represent the euro-denominated price of the pound at maturity.

There are European-style call options written on these three currency pairs at all strikes. It is assumed that the prices of these options are twice differentiable, convex and decreasing with respect to strike. Specifically, there exist two complete sets of dollar-denominated options on the Euro (X) and Pound (Y). There also exists a complete set of cross-rate options, the X-denominated options on the Y. All these options mature at time T. Put options with the same maturity on individual currency pairs are known through the call-put parity.

Within this setup, we make the following assumption:

(A1) There is no arbitrage among all hedging assets.

Since a continuum of call option prices is available and these prices are twice differentiable, Breeden and Litzenberger (1978) have established that the pricing density of Arrow-Debreu claims can be inferred from option prices. Therefore, the available option prices imply that for each of three exchange rates there exists a state price density. Since option prices are twice differentiable, each price density is continuous with respect to strike.

Let $\pi^i(k)$ $(i = X, Y, Z)$ be the price density of an Arrow-Debreu claim that pays 1 unit domestic currency if an exchange rate reaches the level k and zero otherwise. Similarly, let the integrable function $p(x, y)$ be the price density (or pricing function) of a claim that pays \$1 at maturity if $X = x$ and $Y = y$. Since the interest rates in all currencies are zero, the pricing function p has two properties: 1) $\int_{\mathbb{R}_2^+} p(x, y)dxdy = 1$ and 2)$p(x, y) \geq 0$. Let $\mathcal{P}$ be the set of all pricing functions. Assumption $(A1)$ implies that the set $\mathcal{P}$ is not empty.

Lemma 1. *Given the dollar-denominated options on the X and Y and the X-denominated options on the Y, the absence of arbitrage implies the following equalities:*

$$1)\ \pi^X(x) = \int_{\mathbb{R}^+} p(x, y)dy;\ 2)\ \pi^Y(y) = \int_{\mathbb{R}^+} p(x, y)dx;\ 3)\ \pi^Z(z) = \frac{1}{x_0}\Big(\int_{\mathbb{R}^+} xp(x, xz)dx\Big), \tag{2}$$

The following lemma shows that the similar properties in Lemma 1 are maintained if the currency base is changed.

Lemma 2. *Given a pricing function $p \in \mathcal{P}$ that satisfies the conditions in (2), a new pricing function p_X in the currency base X can be derived from the function p in the following way:*

$$p_X(x, y) = \frac{1}{x_0 x} p\Big(\frac{1}{x}, \frac{y}{x}\Big),\ \textit{for } (x, y) \in \mathbb{R}_2^+,$$

such that

$$1)\ \int_{\mathbb{R}^+} p_X(x, y)dy = \frac{\pi^X(\frac{1}{x})}{x_0 x};\ 2)\ \int_{\mathbb{R}^+} p_X(x, y)dx = \pi^Z(y);\ 3)\ \int_{\mathbb{R}^+} xp_X(x, xz)dx = \frac{\pi^Y(z)}{x_0}. \tag{3}$$

Throughout this paper, we are interested in the valuation bounds on a basket call option that delivers the dollar-denominated payoff[3]

$$(\alpha X + \beta Y - \gamma)^+,\ \text{for } (\alpha, \beta, \gamma) \in \mathbb{R}^3. \tag{4}$$

Note that this representation also contains the payoffs of spread options. As far as the valuation bounds on this call is attained, the "put-call" parity relationship for European-style basket options, as pointed out by Hobson, Laurence and Wang (2005a, 2005b) and

[3]The function $(x)^+$ takes the non-negative part of x.

Su (2005), immediately implies that the model-independent valuation bounds on a basket put can be obtained:

$$(\gamma - (\alpha X + \beta Y))^+ = (\alpha X + \beta Y - \gamma)^+ - (\alpha X + \beta Y - \gamma). \tag{5}$$

As for the payoff in (4), state price densities implied from option prices impose restrictions on a set of pricing functions. The possible value range of this payoff is determined by all pricing functions in $\mathcal{P}$ that satisfy the conditions in (2). We now establish valuation bounds on call options in (4).

3 Valuation Bounds and Hedging Portfolios

We seek arbitrage valuation bounds. When dollar-denominated options are traded as hedging instruments, the results in Hobson, Laurence and Wang (2005a, 2005b) show that valuation bounds on the payoff in (4) ($\alpha, \beta, \gamma > 0$) can be attained. We will show the main result that valuation bounds are (much) tightened when cross-rate options are traded as hedging instruments. Valuation bounds are enforced by static portfolios of dollar-denominated options and cross-rate options. Also, the pricing functions that support these valuation bounds are characterized.

We at present only look at upper price bounds, and lower bounds will be discussed later. This section first formulates the valuation problem of a basket option in (4) as an infinite-dimensional LP. This problem is to find the maximum price bound on this basket option within a set of pricing functions. These functions are subject to restrictions imposed by option prices. The dual problem is to search for dominating strategies which enforce upper price bounds. These strategies ensure that an agent who writes this option can put a floor on potential losses.

3.1 Problem Formulation

Consider the basket option that delivers the payoff in (4) at maturity. Its dollar-denominated price is formally expressed as follows:

$$\mathbb{E}_p[(\alpha X + \beta Y - \gamma)^+] = \int_{\mathbb{R}_2^+} [\alpha x + \beta y - \gamma]^+ p(x, y) dx dy, \text{ for } p \in \mathcal{P}. \tag{6}$$

Hence, the price bound on this option is attained by seeking all pricing functions over the entire set $\mathcal{P}$.

To seek the least upper price bound, we express the valuation problem as an LP:

$$\max_{p \in \mathcal{P}} \int_{\mathbb{R}_2^+} [\alpha x + \beta y - \gamma]^+ p(x, y) dx dy \tag{7}$$

s.t.

$$1) \int_{\mathbb{R}^+} p(x, y) dy = \pi^X(x); 2) \int_{\mathbb{R}^+} p(x, y) dx = \pi^Y(y); 3) \int_{\mathbb{R}^+} \frac{x}{x_0} p(x, xz) dx = \pi^Z(z).$$

The initial market prices of options are incorporated into the three constraints. Furthermore, the first two constraints ensure the following

$$\int_{\mathbb{R}_2^+} p(x, y) dx dy = 1.$$

The feasible set of this program is not empty due to assumption $(A1)$. The value of this program is bounded below by zero and from above:

$$\begin{aligned}\mathbb{E}_p[(\alpha X + \beta Y - \gamma)^+] &= \mathbb{E}_p[(\alpha(X - k_1) + \beta(Y - k_2) + (\alpha k_1 + \beta k_2 - \gamma))^+] \\ &\leq \mathbb{E}_p[(\alpha(X - k_1))^+] + \mathbb{E}_p[(\beta(Y - k_2))^+] + \mathbb{E}_p[(\alpha k_1 + \beta k_2 - \gamma)^+] < \infty,\end{aligned} \tag{8}$$

for any strikes $k_1, k_2 \in \mathbb{R}^+$. So the program in (7) must have a solution.

The dual of the problem in (7) is to find the cheapest dominating strategy. Let $\phi = (g(x), h(y), f(z))$ $(x, y, z \in \mathbb{R}^+)$ be a trading strategy such that the functions g, h and f represent the respective components of dollar-denominated and cross-rate options. As a result, the hedging problem may be described as follows:

$$\min_{g,h,f \in \mathbb{R}} \int_{\mathbb{R}^+} g(x)\pi^X(x)dx + \int_{\mathbb{R}^+} h(y)\pi^Y(y)dy + x_0 \int_{\mathbb{R}^+} f(z)\pi^Z(z)dz, \tag{9}$$

s.t.

1′) $g(x) + h(y) + xf(z)\mathbf{1}_{z=y/x} \geq (\alpha x + \beta y - \gamma)^+$ for all $x, y, z \in \mathbb{R}^+$,

where $\mathbf{1}_{(\cdot)}$ is an indicator function. Both programs have the same Lagrangian form. Their equivalence is established through the following result:

Proposition 1 ((Strong Duality)). *Given assumption* $(A1)$*, the values of the primal in (7) and the dual in (9) coincide.*

This strong duality follows from Isii (1963), Gotoh and Konno (2002) and Laurence and Wang (2005). The necessary condition required in Isii's theorem is satisfied in (8):

$$\mathbb{E}_p[(\alpha X + \beta Y - \gamma)^+] < \infty, \text{ for all } p \in \mathcal{P}.$$

Hence, there exists a strategy which involves trading dollar-denominated options and cross-rate options. All the strategies that solve the program in (9) construct a non-empty set $\mathcal{A}$.

Lemma 2 shows that if the currency base is changed, a new pricing function can be constructed from a pricing function that solves the program in (7). This new pricing function also maximizes the basket option's price in the new currency base, and the associated trading strategy is a dominating one.

Proposition 2. *Given the dollar as the currency base, suppose that there exists a pair* (p, ϕ) $(p \in \mathcal{P}, \phi \in \mathcal{A})$ *that supports the market prices of all traded options. If* (p, ϕ) *is optimal for the programs in (7) and (9) respectively, so is* (p_X, ϕ) *based on the currency* X*.*

This proposition establishes the correspondence between pricing and hedging basket options in different currency bases. Dominating strategies do not depend on the choice of numeraire currency. In the next section, we first establish upper bounds on basket options in the case where only dollar-denominated options are tradable. Valuation bounds on currency basket options with two underlying assets may be sought by solving the primal problem where the third restriction in (7) is dropped. Trading strategies that enforce these bounds can be sought by setting $f(z) \equiv 0$ in (9). We then assume that cross-rate options are also traded in markets, and show how these options can tighten valuation bounds.

3.2 Upper Valuation Bounds

When only the dollar-denominated options on the X and Y, the upper price bounds on a basket option with positive parameters $(\alpha, \beta, \gamma > 0)$ can be attained by applying the result in Hobson, Laurence and Wang (2005a). These bounds are enforced by static portfolios of dollar-denominated options.

Proposition 3. *Given a triplet $(\alpha, \beta, \gamma) \in \mathbb{R}_3^+$, the upper price bounds on the option in (4) are enforced by the dollar-denominated options on the X and Y as follows:*

$$\min_{K_a, K_b \geq 0 : \alpha K_a + \beta K_b = \gamma} \alpha \int_{\mathbb{R}^+} (x - K_a)^+ \pi^X(x) dx + \beta \int_{\mathbb{R}^+} (y - K_b)^+ \pi^Y(y) dy. \tag{10}$$

The associated pricing function $p \in \mathcal{P}$ is characterized as follows:

$$p^*(x, y) = \begin{cases} \geq 0, & \textit{if } (x, y) \in (0, K_a^*] \times (0, K_b^*]; \\ \geq 0, & \textit{if } (x, y) \in (K_a^*, \infty) \times (K_b^*, \infty); \\ 0, & \textit{otherwise}, \end{cases} \tag{11}$$

where the strikes K_a^* and K_b^* solve the problem (10).

From Hobson, Laurence and Wang (2005a), the cheapest dominating strategy is sought via a Lagrangian approach. As shown in (10), dominating strategies are to buy call options on the Euro with strike K_a and call options on the Pound with strike K_b. For a basket option in (4) $(\alpha, \beta, \gamma > 0)$, the strikes are chosen so that in the region where both calls are in the money, and two sets of options replicate it exactly. There is no possibility of one option being in the money and the other being out of the money. Since there is no assumption on the behavior of currency prices, these dominating strategies are robust to both model and correlation misspecification.

The dual provides information about the joint distribution of the variables X and Y at maturity. Conditional on the marginal price densities, the joint density in (11) that maximizes the value of the basket option suggests that the variables X and Y are strongly correlated. The left panel in Figure 1 illustrates this pricing function. Since the basket option is an option on a basket of the Euro (X) and Pound (Y), maximizing the correlation between X and Y ensures maximum volatility for the basket and hence maximum value for an option on the basket.

If the X-denominated options on the Y are traded, information embedded in these options tends to restrict the range of correlation between X and Y. The following statement establishes that valuation bounds on basket options are enforced by static portfolios that consist of both dollar-denominated and cross-rate options.

Proposition 4. *Given a triplet $(\alpha, \beta, \gamma) \in \mathbb{R}_3^+$, the upper price bounds on the option in (4) are enforced by the dollar-denominated options on the X and Y and the X-denominated options on the Y as follows:*

$$\begin{aligned} \min_{\lambda, \delta, z_1, z_2, K_i, i=1,2,3,4} & \int_{\mathbb{R}^+} [(\alpha - \lambda)(K_1 - x)^+ + \lambda(x - K_2)^+ \\ & + (\alpha - \lambda)(x - K_3)^+ + \lambda(x - K_4)^+] \pi^X(x) dx \\ & + \int_{\mathbb{R}^+} [(\beta - \delta)(z_1 K_2 - y)^+ + \delta(y - z_2 K_1)^+ \\ & + (\beta - \delta)(y - z_1 K_4)^+ + \delta(y - z_2 K_3)^+] \pi^Y(y) dy \\ & + \int_{\mathbb{R}^+} [-(\beta - \delta)(z_1 - z)^+ + (-\delta)(z - z_2)^+] \pi^Z(z) dz \end{aligned} \tag{12}$$

where

$$1)(\beta-\delta)z_1=\lambda;\delta z_2=\alpha-\lambda,\ \textit{for } 0\le z_1\le\tfrac{\alpha}{\beta}\le z_2 \textit{ and } \lambda\delta\neq 0;$$
$$2)\lambda(K_3-K_2)+\delta(z_1K_4-z_2K_1)=\alpha K_3+\beta z_1K_4-\gamma;$$
$$3)0\le\lambda<\alpha;0\le\delta<\beta;0\le K_1\le K_2\le K_3\le K_4.$$

The associated probability density function $p\in\mathcal{P}$ is characterized as follows:

$$p^*(x,y)=\begin{cases}\ge 0, & \textit{if } (x,y)\in(0,K_1^*]\times[z_2^*K_1^*,z_1^*K_4^*]\cup[K_1^*,K_2^*]\times[z_1^*K_2^*,z_2^*K_1^*];\\ \ge 0, & \textit{if } (x,y)\in[K_2^*,K_3^*]\times(0,z_1^*K_2^*]\cup[K_2^*,K_3^*]\times[z_2^*K_3^*,\infty);\\ \ge 0, & \textit{if } (x,y)\in[K_3^*,K_4^*]\times[z_1^*K_4^*,z_2^*K_3^*]\cup[K_4^*,\infty)\times[z_2^*K_1^*,z_1^*K_4^*];\\ =0, & \textit{otherwise,}\end{cases} \tag{13}$$

where the strikes K_i^* $(i=1,2,3,4),z_1^*,z_2^*$ and real numbers λ^*,δ^* solve the problem (12).

To dominate a basket option, hedging strategies involve eight variables. Four variables K_i $(i=1,2,3,4)$ are used to determine the specific price levels on the Euro (X). These variables together with two variables z_1 and z_2 determine the corresponding price levels on the Pound (Y). These price levels are strikes for buying or selling dollar-denominated options or cross-rate options. The quantity of each hedging instrument is specified by the variables λ and δ. There exists a feasible solution which is identified in (12):

$$\begin{aligned}&z_1\to 0;z_2\to\infty;K_1\to 0;K_4\to\infty;\lambda=0;\delta=0;\\&K_2=K_3=K_a;z_2K_1=z_1K_4=K_b;z_1K_2\to 0;z_2K_3\to\infty,\end{aligned} \tag{14}$$

where $\alpha K_a+\beta K_b=\gamma$. Note that the equalities in the first constraint are valid only for $\lambda\in(0,\alpha)$ and $\delta\in(0,\beta)$, and hence the feasible solution above is consistent with the constraints in (12). This program therefore has a solution.

In the presence of only dollar-denominated options, a bivariate process (X,Y) that maximizes a basket option's price implies the strong dependence between two currency pairs. By incorporating information about cross-rate options, valuation bounds on basket options can be tightened. As a result, the pricing function in (13) that also maximizes the basket option' price indicates that the strong dependence two currency pairs might be unnecessary due to restrictions on correlation between them imposed by cross-rate options. The right panel in Figure 1 illustrates this pricing function.

Through Proposition 3 and 4, the upper bounds on basket options are derived for positive parameters $(\alpha,\beta,\gamma>0)$. However, it is unnecessary to require that a currency basket is constructed by only buying two currencies and selling another. This restriction is relaxed through the following statement.

Proposition 5. *Given a triplet $(\alpha,\beta,\gamma)\in\mathbb{R}_3^-$, the upper price bounds on the option in (4) are enforced by the dollar-denominated options on the X and Y as follows:*

$$\min_{K_a,K_b\ge 0:\alpha K_a+\beta K_b=\gamma}(-\alpha)\int_{\mathbb{R}+}(K_a-x)^+\pi^X(x)dx+(-\beta)\int_{\mathbb{R}+}(K_b-y)^+\pi^Y(y)dy. \tag{15}$$

If the X-denominated options on the Y are traded, the bounds are further enforced as

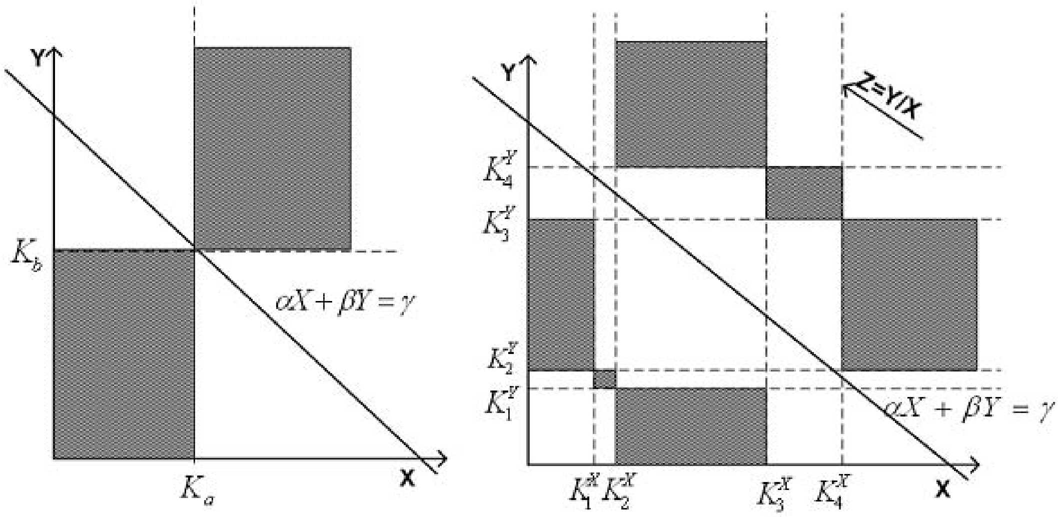

Figure 1: **Joint Density Distributions for Upper Bounds**. The left panel shows a price density (pricing function) $p(x, y)$ that maximizes the price of a basket option in (4) if only dollar-denominated options are traded. The right panel illustrates a price density that maximizes this option's price when cross-rate options are traded as hedging instruments. In the right panel, it has $(K_1^Y, K_2^Y, K_3^Y, K_4^Y) = (z_1K_2^X, z_2K_1^X, z_1K_4^X, z_2K_3^X)$ for $0 < z_1 < z_2$. In both panels, the shaded areas present the regions where the density p is non-negative, while the blank areas indicate zero probabilities.

follows:

$$\begin{aligned}\min_{\lambda,\delta,z_1,z_2,K_i,i=1,2,3,4} \int_{\mathbb{R}+} &[\lambda(K_1 - x)^+ + (-\alpha - \lambda)(K_2 - x)^+ \\ &+ \lambda(K_3 - x)^+ + (-\alpha - \lambda)(x - K_4)^+]\pi^X(x)dx \\ &+ \int_{\mathbb{R}+} [\delta(z_1K_2 - y)^+ + (-\beta - \delta)(z_2K_1 - y)^+ \\ &+ \delta(z_1K_4 - y)^+ + (-\beta - \delta)(z_2K_3 - y)^+]\pi^Y(y)dy \\ &+ \int_{\mathbb{R}+} [(-\delta)(z_1 - z)^+ + (-(-\beta - \delta))(z - z_2)^+]\pi^Z(z)dz\end{aligned} \tag{16}$$

where

$$1)\delta z_1 = (-\alpha) - \lambda; (-\beta - \delta)z_2 = \lambda, \ \textit{if}\ 0 \leq z_1 \leq \tfrac{\alpha}{\beta} \leq z_2 \ \textit{and}\ \lambda\delta \neq 0;$$

$$2)\lambda(K_3 - K_2) + \delta(z_1K_4 - z_2K_1) = \alpha K_2 + \beta z_2K_1 - \gamma;$$

$$3)0 \leq \lambda < (-\alpha); 0 \leq \delta < (-\beta); 0 \leq K_1 \leq K_2 \leq K_3 \leq K_4.$$

The proof of this proposition and the characterization of price functions are accomplished similarly, according to Proposition 3 and 4. In fact, upper valuation bounds on the option in (4) are derived from these results, which will be discussed later.

3.3 Lower Valuation Bounds

We have derived upper valuation bounds on currency basket options. Similarly, lower bounds on these options can be attained by solving LPs. From Hobson, Laurence and Wang (2005b), lower price bounds on basket options with two underlying assets are enforced by the so-called "STP" portfolios that involve calls and puts on individual underlying assets. The associated bivariate processes (X, Y) should be counter-monotonic. Nevertheless, their lower valuation bounds are dependent on the number of disjoint intervals over $\mathbb{R}^+$. We establish a lemma to simplify their result.

Lemma 3. *Suppose that $\mathbb{R}^+$ is partitioned into $(2n+1)$ $(n \geq 1)$ disjoint intervals. Given a triplet $(\alpha, \beta, \gamma) \in \mathbb{R}_3^+$, the lower bounds attained by Hobson, Laurence and Wang (2005b) are the non-increasing functions of partition number (n).*

This lemma shows that the greatest lower bound is determined by setting $n = 1$. As a result, sub-replicating strategies are simplified.

Proposition 6. *1) Given a triplet $(\alpha, \beta, \gamma) \in \mathbb{R}_3^+$, the lower price bounds on the option in (4) are enforced by the dollar-denominated options on the X and Y as follows:*

$$\begin{aligned} \max_{0<K_a^1\leq K_a^2, 0<K_b^1\leq K_b^2} & \int_{\mathbb{R}^+} ((-\alpha)(K_a^1 - x)^+ + \alpha(x - K_a^2)^+)\pi^X(x)dx \\ & + \int_{\mathbb{R}^+} ((-\beta)(K_b^1 - y)^+ + \beta(y - K_b^2)^+)\pi^Y(y)dy, \end{aligned} \tag{17}$$

where $\alpha K_a^1 + \beta K_b^2 = \alpha K_a^2 + \beta K_b^1 = \gamma$.

2) Given a triplet $(\alpha, \beta, \gamma) \in \mathbb{R}_3^-$, the lower price bounds on this option are achieved as follows:

$$\begin{aligned} \max_{0<K_a^1\leq K_X\leq K_a^2, 0<K_b^1\leq K_Y\leq K_b^2} & \int_{\mathbb{R}^+} (\alpha(K_a^1 - x)^+ + (-\alpha)(K_X - x)^+ \\ & + \alpha(x - K_X)^+ + (-\alpha)(x - K_a^2)^+)\pi^X(x)dx \\ & + \int_{\mathbb{R}^+} (\beta(K_b^1 - y)^+ + (-\beta)(K_Y - y)^+ \\ & + \beta(y - K_Y)^+ + (-\beta)(y - K_b^2)^+)\pi^Y(y)dy, \end{aligned} \tag{18}$$

where $\alpha K_a^1 + \beta K_b^2 = \alpha K_a^2 + \beta K_b^1 = \alpha K_X + \beta K_Y = \gamma$.

The associated price density function $p \in \mathcal{P}$ is characterized as follows:

$$p^*(x, y) = \begin{cases} \geq 0, & \textit{if } (x, y) \in (0, \hat{K}_a^1] \times [\hat{K}_b^2, \infty) \textit{ for } |\alpha|x + |\beta|y \geq |\gamma|; \\ \geq 0, & \textit{if } (x, y) \in [\hat{K}_a^1, \hat{K}_a^2] \times [\hat{K}_b^1, \hat{K}_b^2] \textit{ for } |\alpha|x + |\beta|y \leq |\gamma| \ ; \\ \geq 0, & \textit{if } (x, y) \in [\hat{K}_a^2, \infty) \times (0, \hat{K}_b^1] \textit{ for } |\alpha|x + |\beta|y \geq |\gamma|; \\ 0, & \textit{otherwise}, \end{cases} \tag{19}$$

where the strikes $\hat{K}_a^1, \hat{K}_a^2$, $\hat{K}_b^1$ and $\hat{K}_b^2$ solve the problem (17) or (18).

The first part of Proposition 6 directly comes from Lemma 3. Dominating strategies involve short selling puts and long buying calls on the Euro (Pound) with strike K_a^1 (K_b^1) and K_a^2 (K_b^2) so that in the regions both calls and puts are in the money and two sets of options replicate the option exactly. Sub-replicating strategies are slightly different when all the parameters in (4) are negative, and they involve trading more calls and puts. Like dominating strategies specified in Proposition 3, dominated strategies identified here are robust to both model and correlation mis-specification. Meanwhile, the dual provides information about the joint density of the variables X and Y at maturity. In order to minimize the basket option's price, the price density function p in (19) suggests that the process (X, Y) should be counter-monotonic. The left panel of Figure 2 illustrates this pricing function.

Now we derive tight valuation bounds when cross-rate options are traded. These valuation bounds are also enforced by static portfolios that consist of both dollar-denominated options and cross-rate options. The following result establishes these tight valuation bounds.

Proposition 7. *1) Given a triplet $(\alpha, \beta, \gamma) \in \mathbb{R}_3^+$, the lower price bounds on the option in (4) are enforced by the dollar-denominated options on the X and Y and the X-denominated options on the Y as follows:*

$$\begin{aligned}\max_{\lambda,\delta,z_1,z_2,K_i,i=1,2,3,4} & \int_{\mathbb{R}^+} (\lambda(x-K_3)^+ + (\alpha-\lambda)(x-K_4)^+)\pi^X(x)dx \\ & + \int_{\mathbb{R}^+} (\delta(y-z_2K_1)^+ + (\beta-\delta)(y-z_2K_2)^+)\pi^Y(y)dy \\ & + \int_{\mathbb{R}^+} ((-\beta)(z_1-z)^+ + (\beta-\delta)(z-z_2)^+)\pi^Z(z)dz,\end{aligned} \tag{20}$$

where

$$\begin{aligned}&1)\beta z_1 = \lambda-\alpha; (\delta-\beta)z_2 = \alpha, for\ 0 \le z_1 \le z_2;\\ &2)\lambda(K_4-K_3) + \delta(z_2K_2 - z_2K_1) = \alpha K_4 + \beta z_2K_2 - \gamma;\\ &3)\alpha K_1 + \beta z_2 K_1 = \gamma; \alpha K_3 + \beta z_1 K_3 = \gamma;\\ &4)\lambda \ge \alpha; \delta \ge \beta; 0 \le K_1 \le K_2 \le K_3 \le K_4.\end{aligned}$$

2) Given a triplet $(\alpha, \beta, \gamma) \in \mathbb{R}_3^-$, the lower price bounds on this option are enforced as follows:

$$\begin{aligned}\max_{\lambda,\delta,z_1,z_2,K_i,i=1,2,3,4} & \int_{\mathbb{R}^+} ((\lambda-\alpha)(K_3-x)^+ + (-\lambda)(K_4-x)^+)\pi^X(x)dx \\ & + \int_{\mathbb{R}^+} ((\delta-\beta)(z_2K_1-y)^+ + (-\delta)(z_2K_2-y)^+)\pi^Y(y)dy \\ & + \int_{\mathbb{R}^+} (\beta(z_1-z)^+ + (-\delta)(z-z_2)^+)\pi^Z(z)dz,\end{aligned} \tag{21}$$

where

$$\begin{aligned}&1)\beta z_1 = -\lambda; \delta z_2 = -\alpha, for\ 0 \le z_1 \le z_2;\\ &2)\lambda(K_3-K_4) + \delta(z_2K_1 - z_2K_2) = \alpha K_3 + \beta z_2K_1 - \gamma;\\ &3)\alpha K_1 + \beta z_2 K_1 = \gamma; \alpha K_3 + \beta z_1 K_3 = \gamma;\\ &4)\lambda \ge 0; \delta \ge 0; 0 \le K_1 \le K_2 \le K_3 \le K_4.\end{aligned}$$

The associated price function $p \in \mathcal{P}$ is characterized as follows:

$$p^*(x,y) = \begin{cases} \ge 0, & if\ (x,y) \in (0,\hat{K}_3] \times (0, \hat{z}_2\hat{K}_1]\ for\ \frac{y}{x} \in [\hat{z}_1, \hat{z}_2]\ and\ |\alpha|x + |\beta|y \le |\gamma|\ ;\\ \ge 0, & if\ (x,y) \in (0,\hat{K}_2] \times [\hat{z}_2\hat{K}_1, \hat{z}_2\hat{K}_2]\ for\ \frac{y}{x} \ge \hat{z}_2\ and\ |\alpha|x + |\beta|y \ge |\gamma|\ ;\\ \ge 0, & if\ (x,y) \in [\hat{K}_3,\hat{K}_4] \times (0, \hat{z}_1\hat{K}_4]\ for\ \frac{y}{x} \le \hat{z}_1\ and\ |\alpha|x + |\beta|y \ge |\gamma|\ ;\\ \ge 0, & if\ (x,y) \in [\hat{K}_4,\infty) \times [\hat{z}_2\hat{K}_2, \infty)\ for\ \frac{y}{x} \in [\hat{z}_1, \hat{z}_2];\\ 0, & otherwise,\end{cases}$$

where the strikes $\hat{K}_1, \hat{K}_2, \hat{K}_3, \hat{K}_4$, $\hat{z}_1$ and $\hat{z}_2$ solve the problem (20) or (21).

The existence of the strikes that satisfy the conditions in Proposition 7 imply that there at least exits one dominated trading strategy that consists of dollar-denominated options and cross-rate options. This strategy is independent of model specification. Note that the strategies identified in Proposition 6 can be viewed as particular cases of dominated strategies specified in Proposition 7 for $z_1 \to 0$ and $z_2 \to \infty$. Meanwhile, the joint density of the process (X, Y) that minimizes the basket option's price is shown in the right panel of Figure 2.

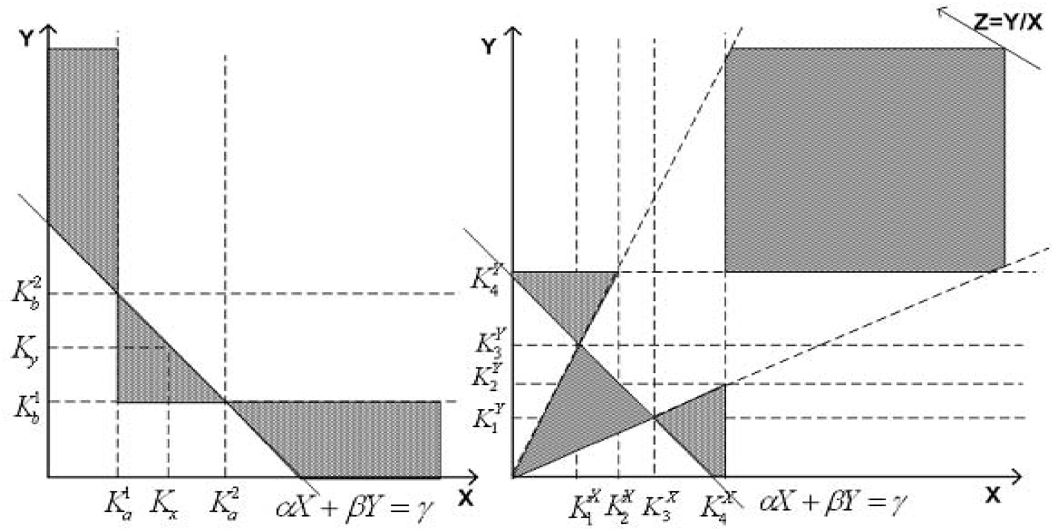

Figure 2: **Joint Density Distributions for Lower Bounds**. The left panel shows a price density (pricing function) $p(x, y)$ that minimizes the price of a basket option in (4) if only dollar-denominated options are traded. The right panel illustrates a price density that minimizes this option's price when cross-rate options are traded as hedging instruments. In the right panel, it has $(K_1^Y, K_2^Y, K_3^Y, K_4^Y) = (z_1 K_3^X, z_1 K_4^X, z_2 K_1^X, z_2 K_2^X)$. In both panels, the shaded areas present the regions where the density p is non-negative, while the blank areas indicate zero probabilities.

Since the strategies identified in Proposition 6 can be viewed as the portfolios without the cross-rate options in an augmented hedging instrument set, the bounds derived from the program in (20) or (21) should not be cheaper than the bounds attained from program in (17) or (18). Furthermore, we now establish the following statement that both upper and lower valuation bounds on basket options with general payoffs are attainable.

Theorem 1. *Given a triplet $(\alpha, \beta, \gamma) \in \mathbb{R}_3$, the upper valuation bounds on the option in (4) can be derived from Proposition 3, 4 and 5, while the lower valuation bounds are attained through Proposition 6 and 7.*

Proof The signs of α, β, γ have the following possible combinations:

$\sharp$	α	β	$(-\gamma)$	$\sharp$	α	β	$(-\gamma)$
1	+	+	+	5	+	-	-
2	+	+	-	6	-	-	+
3	+	-	+	7	-	+	-
4	-	+	+	8	-	-	-

For $\sharp$1 and $\sharp$8, they are degenerate in the sense that the price of the option **b** is always ($\sharp$1) or never ($\sharp$8) in the money. We have sought valuation bounds in $\sharp$2 (Proposition

3 and 4) and ♯6 (Proposition 5). Valuation bounds in ♯3 and ♯4 can be attained from Proposition 3 in the appropriate numeraire currencies. Similarly, Proposition 5 can be applied to ♯7 and ♯8 by changing currency bases. Similarly, the lower valuation bounds then can be attained from Proposition 6 and 7.

4 Numerical Analysis

Given the prices of options on three currency pairs, we investigate arbitrage bounds on basket options, and put the problem into a discrete setup. Within this setup, the prices of call options on all currency pairs are generated and accordingly three price densities are constructed. From these price densities, a numerical procedure for seeking both upper bounds and dominating strategies is proposed. Finally, we qualify the tightness of valuation bounds, and investigate their sensitivity to relevant parameters.

4.1 Model Implementation

Suppose that the variables X and Y take values in the sets

$$\begin{aligned} X &\in \{x_0 u^n | n \in \mathbb{N}\}, \\ Y &\in \{y_0 u^n | n \in \mathbb{N}\}, \end{aligned} \tag{22}$$

for three positive initial numbers $x_0, y_0,\ u > 1$, and $\mathbb{N} = \{-N, \cdots, +N\}$ (an integer number N). The variable Z is determined by $Z = Y/X$.

Now consider a basket option that pays $[\alpha X + \beta Y - \gamma]^+$ dollars at maturity. In this finite-state model, the problems presented in Section 3.1 are naturally expressed as finite-dimensional LPs. Let a $(2N+1)\times(2N+1)$ matrix $P = \{p_{m,n}\}$ represent a pricing function. Let two $(2N+1)$ vectors $\{\pi^X_m\}$ and $\{\pi^Y_n\}$ $(m, n \in [-N, N])$ represent the price densities of Arrow-Debreu claims implied from the dollar-denominated options on the X and Y. To ensure that all three price densities are consistent in scale, a restriction on P is imposed so that $p_{m,n} = 0$ if $|m - n| > N$. For the price density of Arrow-Debreu claims implied from cross-rate options, represented by a vector $\{\pi^Z_j\}$, this restriction equivalently states $\pi^Z_{j=m-n} \geq 0$ for $|m - n| \leq N$ and zero otherwise.

To seek the least upper bound on this basket option, the primal problem in (7) can be reexpressed as a finite LP

$$\text{(LP1) find the } (2N+1)\times(2N+1) \text{ matrix } P \text{ so that}$$
$$\max_P \sum_{m,n} (\alpha x_0 u^m + \beta y_0 u^n - \gamma)^+ p_{m,n},$$

s.t.

1) $\sum_{n=-N}^{N} p_{m,n} = \pi^X_m$ for $m \in [-N, N]$;

2) $\sum_{m=-N}^{N} p_{m,n} = \pi^Y_n$ for $n \in [-N, N]$;

3) $\sum_{m=\max(0,-j)-N}^{\min(0,-j)+N} u^m p_{m,m+j} = \pi^Z_j$ for $j \in [-N, N]$;

4) $p_{m,n} \geq 0$ for all m and n.

Assumption $(A1)$ implies that there exists a price density P consistent with all option prices. So the solution set for LP1 is not empty. This program is bounded below by zero and above by $(\alpha x_0 u^N + \beta y_0 u^N - \gamma)^+$. Therefore, this program must have a solution.

Consider a strategy $\phi = (G, H, F)$ whose three components represent trading positions in currency options. Each component is represented by a $(2N + 1)$ vector. To seek the cheapest dominating strategy, the hedging problem in (9) is reexpressed as follows

$$\text{(LP2) find the } (2N+1) \times 1 \text{ vectors } G,\ H \text{ and } F \text{ so that}$$
$$\min_{G,H,F} \sum_m g_m \pi_m^X + \sum_n h_n \pi_n^Y + \sum_j f_j \pi_j^Z,$$

s.t.

1′) $g_m + h_n + u^m f_j \mathbf{1}_{j=n-m} \geq (\alpha x_0 u^m + \beta y_0 u^n - \gamma)^+$ for all m, n, j;

2′) $g_m, h_n, f_j \in \mathbb{R}$ for all m, n, j.

Since the program LP1 has a solution, the LP Duality Theorem implies that the program LP2 must also have a solution.

4.2 Numerical Results

To attain insights into the tightness of arbitrage bounds, we use a trinomial-tree method to simulate the dynamics of the Euro (X) and Pound (Y) exchange rates against the U.S. dollar. These two exchange rate processes are correlated with coefficient ρ over time, and have an identical annual implied volatility, e.g., $\sigma^X = \sigma^Y = \sigma$. Both the domestic and foreign interest rate are zero.

For the sets in (22), the levels of the rate X (Y) are bounded in a range of $[x_0 u^{-J}, x_0 u^J]$ $([y_0 u^{-J}, y_0 u^J])$ where $0 < J < N$. In this way, there are absorbing boundaries imposed on rate levels for computational convenience. For a large J, these boundaries have no significant impact on numerical analysis.

All European option prices on the X and Y are separately generated using the trinomial tree method over $N = 40$ time steps. The price densities π^X and π^Y are calculated from these European options.[4] By applying Boyle (1988)'s numerical procedure, we generate a (feasible) price density P which is consistent with initial option prices on the X and Y. Finally, the price density π^Z is constructed from the price density P.[5]

All relevant parameters are set by

$$(x_0, y_0, t^*, \sigma, \rho, u, N, J) = (1.6, 2.5, 1, 0.42, 0.85, 1.12, 40, 30).$$

The correlation parameter ρ is positive, as both the Pound and Euro exchange rates against the U.S. dollar are usually positively correlated. The left panel in Figure 3 illustrates the initial price density P generated by Boyle's procedure. Each contour line represents the positive prices of this density function. The right panel in Figure 3 reports the price densities of Arrow-Debreu claims on three exchange rates.

[4] For a set of option prices $C(K)$, the price density π is calculated as follows:

$$\pi(K) = \frac{C(Ku, t) - (1+u)C(K) + uC(K/u)}{Ku - K},$$

for each K in the set.

[5] The density π^Z is calculated by $\pi_j^Z = \sum_{m=\max(0,-j)-J}^{\min(0,-j)+J} u^m p_{m,m+j}$ for $j \in [-J, J]$.

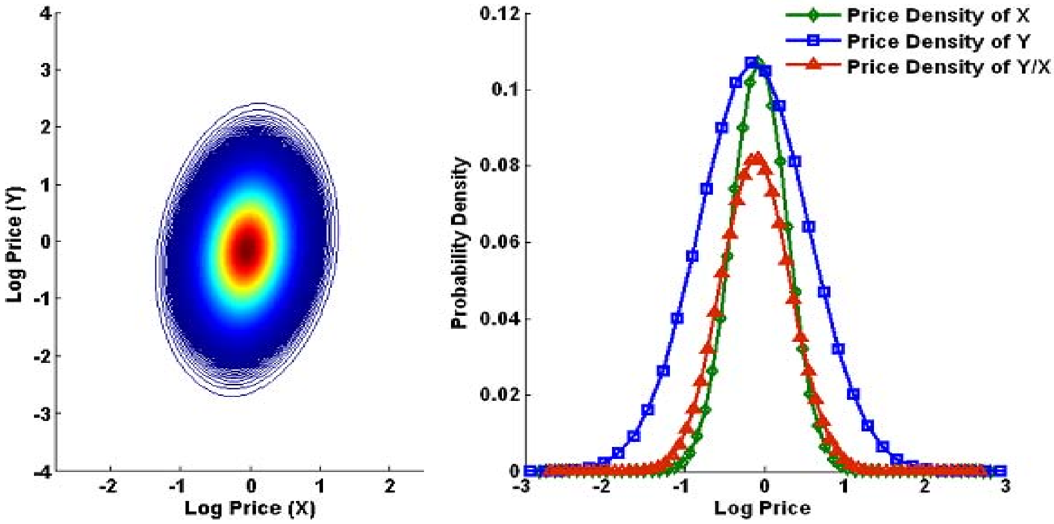

Figure 3: **Initial Probability Densities**. The time-zero prices of the exchange rates X and Y are $x_0 = 1.6$ and $y_0 = 2.5$. The implied volatilities of the processes X and Y are 42%. Both the domestic rate and foreign rate are zero. The marginal densities π^X and π^Y are attained within a price mesh of $u = 1.12$ and $J = 30$. The marginal density π^Z is derived from the joint density that is calculated using the correlation coefficient $\rho = 0.20$. In the left panel, each contour line represents the positive prices of the initial joint density P.

The initial price density P yields a price for a basket option. In order to measure the tightness of arbitrage bounds, this price can be viewed as a benchmark price, denoted as V_{bs}. Let $\overline{\mathcal{B}}_h$ and $\underline{\mathcal{B}}_h$ stand for the upper and lower valuation bounds on this basket option, enforced by the option prices on the X and Y. Let $\overline{\mathcal{B}}_t$ and $\underline{\mathcal{B}}_t$ be valuation bounds if cross-rate options are traded as hedging instruments. These new instruments would tighten arbitrage bounds. To gauge the magnitude of the tightness, we consider the following measure:

$$\epsilon = 1 - \frac{\overline{\mathcal{B}}_t - \underline{\mathcal{B}}_t}{\overline{\mathcal{B}}_h - \underline{\mathcal{B}}_h}. \tag{23}$$

4.2.1 Valuation Bounds and Hedging Strategies

Now consider a basket option that pays $[1.2X + 0.9Y - 3.8]^+$ dollars after one year. Given the different sets of hedging instruments, the price functions that deliver the valuation bounds on this option have been characterized in Figure 1 and 2. Accordingly, the valuation bounds on this option are attained as follows:

$$(\underline{\mathcal{B}}_h, \underline{\mathcal{B}}_t, V_{bs}, \overline{\mathcal{B}}_t, \overline{\mathcal{B}}_h) = (0.38, 0.52, 0.72, 0.83, 0.86).$$

In terms of tightness, the bounds are substantially improved by $\epsilon = 36\%$ when information about the prices of cross-rate options is incorporated.

We further investigate the associated hedging strategies that enforce these valuation bounds. First, the dominating strategy simply involves buying long 1.2 calls on the X with strike $K_a = 1.42$ and buying long 0.9 calls on the Y with strike $K_b = 2.23$. This strategy delivers the bound $\overline{\mathcal{B}}_h$. To attain the tight upper valuation bound $\overline{\mathcal{B}}_t$, the dominating strategy involves buying dollar-denominated puts and calls on the X and Y and selling puts and calls on the Z:

- buying an option portfolio on the X that consists of long 0.56 puts with strike 1.02, long $0.64, 0.56, 0.64$ calls with strikes $1.28, 1.79, 2.25$;

- buying an option portfolio on the Y that consists of long 0.57 puts with strike 1.59, long $0.33, 0.57, 0.33$ calls with strikes $1.78, 2.80, 3.14$;

- selling an option portfolio on the Z that consists of short 0.57 puts with strike 1.24 and short 0.33 calls with strike 1.75.

As a result, this strategy is cheaper by about 0.04 dollars.

To enforce the bound $\underline{\mathcal{B}}_h$, the sub-replicating strategy involves selling short 1.2 puts and buying long 1.2 calls on the X with strikes $K_a^1 = 1.42$ and $K_a^2 = 2.01$, and selling short 0.9 puts and buying long 0.9 calls on the Y with strikes $K_b^1 = 1.99$ and $K_b^2 = 2.80$. The dominated strategy that delivers the bound $\underline{\mathcal{B}}_t$ can be described as follows:

- buying an option portfolio on the X that consists of long 1.52 puts with strike 1.72 short 0.32 calls with strike 2.24;

- buying an option portfolio on the Y that consists of long 1.44 puts with strike 2.23, short 0.54 calls with strike 3.93;

- selling a portfolio of the cross-rate options on the Z that consists of short 0.90 puts with strike 0.88 and short 0.54 calls with strike 2.45.

Equivalently, this strategy improves the lower bound by about 0.14 dollars.

4.2.2 Sensitivity of Valuation Bounds

As shown in Figure 3, an appropriate J is chosen so that three price densities are close to zero when exchange rate levels reach the absorbing boundaries. In this way, valuation bounds on basket options are relatively independent of the number of price levels (N). We now investigate the sensitivity of valuation bounds by varying the jump size (u), as reported in Table 1. In terms of the measure in (23), all the numbers indicate that the price bounds on two options are significantly tightened by incorporating price information about cross-rate options. These numbers also show that these valuation bounds are relatively robust to changes in the jump size u. The price bounds on the first (second) option can at least be improved by an average of 36% (32%).

u	$[1.2X + 0.9Y - 3.8]^+$						$[-X - Y + 4.8]^+$					
	$\underline{\mathcal{B}}_h$	$\underline{\mathcal{B}}_t$	V_{bs}	$\overline{\mathcal{B}}_t$	$\overline{\mathcal{B}}_h$	ϵ	$\underline{\mathcal{B}}_h$	$\underline{\mathcal{B}}_t$	V_{bs}	$\overline{\mathcal{B}}_t$	$\overline{\mathcal{B}}_h$	ϵ
1.12	0.386	0.527	0.718	0.832	0.863	36.0%	0.782	0.874	0.994	1.117	1.143	32.2%
1.14	0.389	0.534	0.717	0.825	0.862	38.6%	0.783	0.875	0.993	1.115	1.139	32.5%
1.16	0.388	0.538	0.714	0.825	0.863	39.5%	0.780	0.873	0.991	1.111	1.134	32.6%
1.18	0.390	0.583	0.712	0.818	0.863	50.4%	0.783	0.871	0.987	1.106	1.131	32.4%
1.20	0.371	0.577	0.709	0.815	0.865	51.8%	0.784	0.876	0.985	1.105	1.134	34.4%

Table 1: **Sensitivity of Valuation Bounds Against Jump Size.** The time-zero prices of the exchange rates X and Y are 1.6 and 2.5. The implied volatilities of the processes X and Y are 42%, and two processes are correlated with $\rho = 0.20$. Both the domestic and foreign interest rate are zero. All prices are attained within a price mesh of $J = 30$. Two basket options mature at $T = 1$ (year).

In order to investigate the sensitivity of valuation bounds to changes in correlation between the Euro and Pound exchange rates, we assume that the coefficient ρ varies in the range of $[-1, 1]$. The arbitrage bound $\mathcal{B}_h$ (either $\underline{\mathcal{B}}_h$ or $\overline{\mathcal{B}}_h$) is independent of correlation coefficient, as this bound is enforced by a portfolio of dollar-denominated options. Nevertheless, changes in correlation coefficient have impact on the valuation bound $\mathcal{B}_t$ (either $\underline{\mathcal{B}}_t$ or $\overline{\mathcal{B}}_t$), as this bound depends on the density π^Z that is constructed from an initial price density P. For each ρ, we use Boyle's procedure to generate a new price density P.

ρ	$[1.2X + 0.9Y - 3.8]^+$						$[-X - Y + 4.8]^+$					
	$\underline{\mathcal{B}}_h$	$\underline{\mathcal{B}}_t$	V_{bs}	$\overline{\mathcal{B}}_t$	$\overline{\mathcal{B}}_h$	ϵ	$\underline{\mathcal{B}}_h$	$\underline{\mathcal{B}}_t$	V_{bs}	$\overline{\mathcal{B}}_t$	$\overline{\mathcal{B}}_h$	ϵ
-0.80	0.387	0.392	0.483	0.519	0.862	73.3%	0.783	0.793	0.805	0.860	1.142	81.2%
-0.60	0.386	0.401	0.545	0.608	0.861	56.4%	0.783	0.802	0.845	0.924	1.142	66.1%
-0.40	0.386	0.425	0.597	0.678	0.860	46.5%	0.783	0.815	0.887	0.981	1.142	53.7%
-0.20	0.386	0.453	0.644	0.735	0.861	40.5%	0.783	0.833	0.927	1.035	1.142	44.0%
0	0.380	0.500	0.706	0.817	0.860	35.0%	0.782	0.852	0.966	1.084	1.142	35.5%
0.20	0.388	0.530	0.717	0.821	0.860	38.3%	0.782	0.874	0.994	1.116	1.142	32.7%
0.40	0.388	0.584	0.757	0.859	0.860	41.7%	0.782	0.910	1.033	1.142	1.142	35.7%
0.60	0.383	0.652	0.793	0.860	0.860	55.4%	0.782	0.961	1.069	1.142	1.142	49.9%
0.80	0.388	0.741	0.829	0.860	0.860	74.5%	0.782	1.023	1.105	1.142	1.142	67.2%

Table 2: **Sensitivity of Valuation Bounds Against Correlation Coefficient.** The time-zero prices of the exchange rates X and Y are 1.6 and 2.5. The implied volatilities of the processes X and Y are 42%. Both the domestic and foreign interest rate are zero. All prices are attained within a price mesh of $u = 1.12$ and $J = 30$. Two basket options mature at $T = 1$ (year).

Table 2 presents the sensitivity of valuation bounds when correlation coefficient varies. The bounds $\mathcal{B}_t$ on two options increase as the coefficient ρ increases. In particular, the bounds $\overline{\mathcal{B}}_t$ approach $\overline{\mathcal{B}}_h$ when the coefficient ρ increases to 1. However, the tightness of valuation bounds decreases as the the coefficient ρ increases from -0.8 to 0 or decreases from 0.8 to 0. For $\rho = -0.8$, the valuation bounds on the first (second) option is substantially tightened by about 73.3% (81.2%). Similarly, the bounds on these two options are also greatly tightened for $\rho = 0.8$, compared with the tightness of bounds at $\rho = 0$ (35.0% and 35.5% respectively).

Since the dependence between the Euro (X) and Pound (Y) has impact on the joint price density P, the construction of the density π^Z means that increasing (decreasing) ρ leads to increases (decreases) in the kurtosis of the density π^Z. For a large ρ (close to 1), the density π^Z with high kurtosis implies that cross-rate options have limited impact on the tightness of upper bounds but significantly improve lower bounds, as shown in Table 2. This impact on both lower and upper price bounds may be substantial when the coefficient ρ decreases to -1, as there are more Arrow-Debreu securities with non-zero prices for trading.

T (year)	$[1.2X + 0.9Y - 3.8]^+$						$[-X - Y + 4.8]^+$					
	$\underline{\mathcal{B}}_h$	$\underline{\mathcal{B}}_t$	V_{bs}	$\overline{\mathcal{B}}_t$	$\overline{\mathcal{B}}_h$	ϵ	$\underline{\mathcal{B}}_h$	$\underline{\mathcal{B}}_t$	V_{bs}	$\overline{\mathcal{B}}_t$	$\overline{\mathcal{B}}_h$	ϵ
0.50	0.370	0.449	0.575	0.655	0.674	32.3%	0.725	0.772	0.848	0.938	0.947	25.3%
0.75	0.376	0.484	0.651	0.751	0.775	33.0%	0.751	0.823	0.925	1.035	1.052	29.7%
1.00	0.388	0.531	0.717	0.821	0.860	38.3%	0.782	0.874	0.994	1.116	1.114	32.7%
1.25	0.391	0.571	0.777	0.898	0.934	39.6%	0.815	0.921	1.057	1.191	1.222	33.9%
1.50	0.426	0.615	0.831	0.965	1.006	39.6%	0.849	0.967	1.145	1.257	1.295	35.1%

Table 3: **Sensitivity of Valuation Bounds Against Maturity.** The time-zero prices of the exchange rates X and Y are 1.6 and 2.5. The implied volatilities of the processes X and Y are 42%, and two processes are correlated with $\rho = 0.20$. Both the domestic and foreign interest rate are zero. All prices are attained within a price mesh of $u = 1.12$ and $J = 30$.

We further examine the sensitivity of price bounds against other parameters (i.e., maturity, implied volatility and non-zero yield rate). Table 3 reports the sensitivity of valuation bounds to different maturities. In term of magnitude, the price bounds on the first (second) option are increasingly tightened from 32.3% (25.3%) to 39.6% (35.1%) as maturity increases from 6 months to 15 months. The similar result is observed in Table

σ	$[1.2X+0.9Y-3.8]^+$						$[-X-Y+4.8]^+$					
	$\underline{\mathcal{B}}_h$	$\underline{\mathcal{B}}_t$	V_{bs}	$\overline{\mathcal{B}}_t$	$\overline{\mathcal{B}}_h$	ϵ	$\underline{\mathcal{B}}_h$	$\underline{\mathcal{B}}_t$	V_{bs}	$\overline{\mathcal{B}}_t$	$\overline{\mathcal{B}}_h$	ϵ
25%	0.371	0.427	0.523	0.592	0.606	29.3%	0.712	0.743	0.799	0.873	0.878	21.5%
30%	0.370	0.451	0.578	0.659	0.679	32.5%	0.726	0.774	0.852	0.942	0.952	25.6%
35%	0.372	0.481	0.635	0.732	0.754	34.3%	0.745	0.812	0.909	1.012	1.030	29.9%
40%	0.384	0.511	0.694	0.801	0.829	35.1%	0.771	0.855	0.969	1.088	1.109	31.3%
45%	0.391	0.551	0.754	0.869	0.908	38.5%	0.802	0.902	1.032	1.161	1.190	33.5%

Table 4: **Sensitivity of Valuation Bounds Against Implied Volatility.** The time-zero prices of the exchange rates X and Y are 1.6 and 2.5. The underlying processes X and Y are correlated with $\rho = 0.20$. Both the domestic and foreign interest rate are zero. All prices are attained within a price mesh of $u = 1.12$ and $J = 30$. Two basket options mature at $T = 1$ (year).

Yield Rate	$[1.2X+0.9Y-3.8]^+$						$[-X-Y+4.8]^+$					
	$\underline{\mathcal{B}}_h$	$\underline{\mathcal{B}}_t$	V_{bs}	$\overline{\mathcal{B}}_t$	$\overline{\mathcal{B}}_h$	ϵ	$\underline{\mathcal{B}}_h$	$\underline{\mathcal{B}}_t$	V_{bs}	$\overline{\mathcal{B}}_t$	$\overline{\mathcal{B}}_h$	ϵ
-10%	0.157	0.313	0.464	0.564	0.598	43.1%	1.108	1.153	1.234	1.348	1.353	20.4%
-5%	0.237	0.405	0.575	0.680	0.714	42.1%	0.938	1.003	1.103	1.222	1.234	26.7%
0%	0.380	0.520	0.704	0.805	0.843	38.3%	0.766	0.856	0.974	1.094	1.119	32.7%
+5%	0.575	0.656	0.849	0.969	0.987	23.9%	0.596	0.713	0.851	0.973	1.007	36.9%
+10%	0.795	0.819	1.015	1.135	1.151	11.0%	0.429	0.581	0.735	0.857	0.901	41.3%

Table 5: **Sensitivity of Valuation Bounds Against Yield Rate.** The time-zero prices of the exchange rates X and Y are 1.6 and 2.5. The implied volatilities of the processes X and Y are 42%, and two processes are correlated with $\rho = 0.20$. The yield rate (Δ) is the difference between the domestic risk-free interest rate ($r_d^\$$) and foreign interest rate (r_f), i.e., $\Delta = r_d^\$ - r_f$. The domestic interest rate is set by 2%, and the foreign interest rates for the Euro (X) and Pound (Y) are assumed to be identical. All prices are attained within a price mesh of $u = 1.12$ and $J = 30$. Two basket options mature at $T = 1$ (year), and thus the discount factor is expressed as $DF = e^{-r_d^\$ T} = 0.98$.

4. The price bounds on the first (second) option are increasingly improved from 29.3% (21.5%) to 38.5% (33.5%) as volatility level increases by 80%.

Table 5 shows that changing interest rates has different impact on two basket options. The tightness of the price bounds on the first option ($\alpha, \beta, \gamma > 0$) is decreasing as the foreign interest rate r_f decreases, while the bounds on the second option ($\alpha, \beta, \gamma < 0$) are increasingly improved. Overall, the valuation bounds on two options can be tightened by cross-rate options for different interest rate levels. The tightness of the price bounds on the first (second) option has an average of about 31.7% (31.5%) when the domestic interest rate is set as 2% and the foreign interest rate varies in a range of $[-8\%, 12\%]$.

5 Conclusions

Basket options are traded as alternative instruments to manage risk exposure in multiple underlying assets in an effective way. Despite their attractive features, pricing basket options poses a challenge to practitioners. Through this paper, we have studied the problem of valuing currency basket options. Instead of precise prices, we derive arbitrage valuation bounds on these options that are robust to the choice of specific pricing models.

We identify a collection of tradable options on individual currency pairs as hedging instruments. Three currencies, the Euro, British Pound and U.S. dollar are considered. We emphasize the role of price information about cross-rate options in pricing basket options. More specifically, arbitrage bounds are derived from portfolios that involve only dollar-denominated options. If cross-rate options are additionally traded, price bounds on basket options can be further tightened, and are enforced by static portfolios of both dollar-denominated options and cross-rate options. The tightness of these bounds highlights the informational content embedded in cross-rate options, implying that the incorporation of the information about these options may be helpful to reduces arbitrage opportunities imposed on currency basket options to a significant extent.

We have derived both upper and lower bounds on currency basket options. These valuation bounds are enforced by static hedging strategies that are constructed from available hedging instruments, e.g., the options on individual currency pairs. In particular, it is found that the lower bounds derived by Hobson, Laurence and Wang (2005b) have the reduced form. Meanwhile, these strategies are associated with the joint densities of underlying currencies. These densities then characterize those pricing models that produce robust valuation bounds on currency basket options.

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A Proof of Lemma 1

Proof Let $C(K)$ be the market price of an option with payoff $(S-K)^+$ for $K \in \mathbb{R}^+$. Since a continuum of option prices is available at inception and these prices are twice differentiable, the price density of Arrow-Debreu claims is known from Breeden and Litzenberger (1978):

$$\pi(k) = \frac{\partial^2 C(K)}{\partial K^2}|_{K=k}, \tag{24}$$

for each strike k. Given the prices of the dollar-denominated options on the X and Y, and the X-denominated options on the Y, there exists a price density for each currency pair.

Let $p(x,y)$ represent the price density of a claim that pays \$1 if $X = x$ and $Y = y$. The non-arbitrage argument in assumption $(A1)$ implies that there must exist a risk-neutral probability measure μ such that

$$\pi^X(x) = \mathbb{E}^\mu[1_{X=x}] = \mathbb{E}^\mu[\int_{\mathbb{R}+} 1_{X=x}1_{Y=y}dy] = \int_{\mathbb{R}+} \mathbb{E}^\mu[1_{X=x}1_{Y=y}]dy = \int_{\mathbb{R}+} p(x,y)dy, \tag{25}$$

given the dollar as the numeraire currency. Similarly, we also have

$$\pi^Y(y) = \mathbb{E}^\mu[1_{Y=y}] = \int_{\mathbb{R}+} p(x,y)dx. \tag{26}$$

Since the X-denominated price of a claim that pays 1 unit X if $X = x$ and $Y = xz$ has the density $\pi^Z(z)$, it is equivalent to stating that

- the time-0 price density of this claim is $x_0\pi^Z(z)$;
- the time-T payoff of this claim is $\int_{\mathbb{R}+} x1_{X=x}1_{Y=xz}dx$,

if the U.S. dollar is the base currency. This statement leads to the third equality

$$\pi^Z(z) = \frac{1}{x_0}\mathbb{E}^\mu[\int_{\mathbb{R}+} x1_{X=x}1_{Y=xz}dx] = \frac{1}{x_0}\int_{\mathbb{R}+} xp(x, xz)dx. \tag{27}$$

B Proof of Lemma 2

Proof Given the currency X as the base, let the function $p_X(x, y)$ be the price density of a claim that pays 1 unit X if $1/X = x$ and $Y/X = y$. Equivalently, this claim pays $1/x$ dollars, and its dollar-denominated price has a density $x_0p_X(x, y)$. Recall that the function $p(1/x, y/x)$ is the dollar-denominated price density of a claim that pays one dollar if $X = 1/x$ and $Y = y/x$. To avoid arbitrage, we then have

$$x_0p_X(x, y) = \frac{1}{x}p(\frac{1}{x}, \frac{y}{x}), \tag{28}$$

which immediately results in the first equality

$$\int_{\mathbb{R}+} p_X(x, y)dy = \int_{\mathbb{R}+} \frac{1}{x_0x}p(\frac{1}{x}, \frac{y}{x})d(\frac{y}{x}) = \frac{1}{x_0x}\pi^X(\frac{1}{x}). \tag{29}$$

This equality states that the dollar-denominated price density of a claim that pays 1 unit X if $1/X = x$ is equal to $\pi^X(1/x)/(x_0x)$.

Also, it is easy to derive the following result:

$$\int_{\mathbb{R}+} p_X(x, y)dx = \pi^Z(y), \tag{30}$$

which determines the X-denominated price density of a claim that pays 1 unit X if $Y/X = y$.

To derive the third equality, consider a claim that pays 1 unit X if $1/X = x$ and $Y/X = xz$ for $z \in \mathbb{R}^+$. Its X-denominated price density is $p_X(x, xz)$. Meanwhile, it is known that the dollar-denominated price density of a claim that pays \$1 if $Y = z$ is $\pi^Y(z)$ such that

$$x_0\int_{\mathbb{R}+} xp_X(x, xz)dx = \int_{\mathbb{R}+} p(\frac{1}{x}, z)d(\frac{1}{x}) = \pi^Y(z), \tag{31}$$

which yields the third equality.

C Proof of Proposition 2

Proof Given the dollar as the currency base, suppose that there exists a pair of (p, ϕ) ($p \in \mathcal{P}, \phi \in \mathcal{A}$) that solves two programs in (7) and (9). From Lemma 2, a price density in the currency base X can be determined as follows:

$$p_X(x, y) = \frac{1}{x_0x}p(\frac{1}{x}, \frac{y}{x}), \text{ for } (x, y) \in \mathbb{R}_2^+. \tag{32}$$

Then if the function p maximizes the dollar-denominated price of a basket option, the new function p_X also maximizes the X-denominated price of this option in order to avoid arbitrage.

To see that the strategy ϕ dominates the basket option based on the currency X, consider its X-denominated payoff:

$$[\alpha + \beta y - \gamma x]^+, \text{ for } (x, y) \in \mathbb{R}_2^+. \tag{33}$$

As the dual of pricing, hedging this payoff would involve a portfolio of both the dollar-denominated options on the X and Y and the X-denominated options on the Y:

$$g(\frac{1}{x})x + xh(w)1_{w=y/x} + f(y) \geq [\alpha + \beta y - \gamma x]^+, \tag{34}$$

which is equivalent to holding $g(\frac{1}{x})$ claims if $1/X = x$, $h(w)$ claims if $Y = w$ and $f(y)$ claims if $Y/X = y$ at inception. This inequality may be expressed as in dollars

$$g(\frac{1}{x}) + h(w) + \frac{1}{x}f(y)1_{y=xw} \geq [\alpha\frac{1}{x} + \beta\frac{y}{x} - \gamma]^+. \tag{35}$$

This is identical to the expression by setting $x = \frac{1}{\widehat{x}}$

$$g(\widehat{x}) + h(w) + \widehat{x}f(y)1_{y=w/\widehat{x}} \geq [\alpha\widehat{x} + \beta w - \gamma]^+. \tag{36}$$

Since the strategy $\phi = (g, h, f)$ solves the dual problem in (9), the inequality above implies that this strategy also dominates the basket option in the currency base X.

D Proof of Proposition 3

Proof To derive upper bounds, consider the dollar-denominated payoff of a basket option in (4):

$$(\alpha X + \beta Y - \gamma)^+ \leq \alpha(X - \frac{\gamma\lambda_1}{\alpha})^+ + \beta(Y - \frac{\gamma\lambda_2}{\beta})^+, \text{ for } (\alpha, \beta, \gamma) \in \mathbb{R}_3^+, \tag{37}$$

for $\lambda_1, \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 = 1$.

By setting $K_a = \frac{\gamma\lambda_1}{\alpha} \geq 0$ and $K_b = \frac{\gamma\lambda_2}{\beta} \geq 0$, the upper bound on this basket option is determined as follows:

$$\overline{\mathcal{B}} = \min_{K_a, K_b \geq 0: \alpha K_a + \beta K_b = \gamma} \alpha \int_{\mathbb{R}+} (x - K_a)^+ \pi^X(x)dx + \beta \int_{\mathbb{R}+} (y - K_b)^+ \pi^Y(y)dy. \tag{38}$$

Two strikes K_a and K_b are determined by λ_1 and λ_2. The solution set for this program is not empty. This program is bounded above, and so it must have a solution. Let K_a^* and K_b^* be the optimal solutions for this program such that

$$\overline{\mathcal{B}} = \alpha \int_{\mathbb{R}+} (x - K_a^*)^+ \pi^X(x)dx + \beta \int_{\mathbb{R}+} (y - K_b^*)^+ \pi^Y(y)dy. \tag{39}$$

To see that there exists a density function p that supports this bound, define two separated sets

$$\begin{aligned} A &= (0, K_a^*] \times (0, K_b^*] \cup (K_a^*, \infty) \times (K_b^*, \infty); \\ B &= \mathbb{R}_2^+ \backslash A. \end{aligned} \tag{40}$$

Consider a candidate density function:

$$p^*(x, y) = \begin{cases} \geq 0, & \text{if } (x, y) \in A; \\ 0, & \text{if } (x, y) \in B, \end{cases} \tag{41}$$

such that $\int_{\mathbb{R}+} p^*(x, y)dy = \pi^X(x)$ and $\int_{\mathbb{R}+} p^*(x, y)dx = \pi^Y(y)$.

To ensure the existence of such a density function, consider the construction of a process (X, Y) so that the variable Y is an increasing function of the X. Given any $\nu \sim U[0, 1]$, there exists a real number vector $(\bar{x}, \bar{y}) \in \mathbb{R}_2^+$ and the random variable vector (X, Y) so that

$$prob(X \leq \bar{x}) = prob(Y \leq \bar{y}) = \nu. \tag{42}$$

A bivariate process (X, Y) is constructed through the inverse function, and so this process is comonotonic. This process implies that:

i) the events $(X \leq \bar{x})$ and $(Y \leq \bar{y})$ are mutually compatible each other and so do the events $(X > \bar{x})$ and $(Y > \bar{y})$;

ii) the events $(X > \bar{x})$ and $(Y \leq \bar{y})$ are mutually exclusive each other and so do the events $(X \leq \bar{x})$ and $(Y > \bar{y})$.

By setting $\bar{x} = K_a^*$ and $\bar{y} = K_b^*$, we have the following equality in the regions where the events $(X > \bar{x}, Y > \bar{y})$ or $(X \leq \bar{x}, Y \leq \bar{y})$ occur:

$$(\alpha X + \beta Y - \gamma)^+ = \alpha(X - K_a^*)^+ + \beta(Y - K_b^*)^+. \tag{43}$$

In the regions where the events $(X \leq \bar{x}, Y > \bar{y})$ or $(X > \bar{x}, Y \leq \bar{y})$ occur, we have the following inequality:

$$(\alpha X + \beta Y - \gamma)^+ < \alpha(X - K_a^*)^+ + \beta(Y - K_b^*)^+. \tag{44}$$

For the candidate density function p^*, the price of the basket option can be expressed as follows:

$$\mathbb{E}_{p^*}[(\alpha x + \beta y - \gamma)^+] = \alpha \int_{\mathbb{R}^+} (x - K_a^*)^+ \pi^X(x)dx + \beta \int_{\mathbb{R}^+} (y - K_b^*)^+ \pi^Y(y)dy = \overline{\mathcal{B}}. \tag{45}$$

E Proof of Proposition 4

Proof 1) We first set up a class of dominating hedging portfolios. Eight price levels are chosen from the X and Y

$$\begin{aligned} &0 < K_1 < K_2 < K_3 < K_4; \\ &0 < K_1^b < K_2^b < K_3^b < K_4^b, \end{aligned} \tag{46}$$

so that these prices determine two price levels on the Z:

$$z_1 = \frac{K_1^b}{K_2} = \frac{K_3^b}{K_4}, z_2 = \frac{K_2^b}{K_1} = \frac{K_4^b}{K_3}. \tag{47}$$

To dominate the payoff of a basket option in (4), consider the strategy ϕ which consists of three components:

i) the holdings of the dollar-denominated options on the X are

- long $(\alpha - \lambda)$ puts at strike K_1;
- long λ calls at strike K_2;
- long $(\alpha - \lambda)$ calls at strike K_3;
- long λ calls at strike K_4;

ii) the holdings of the dollar-denominated options on the Y are

- long $(\beta - \delta)$ puts at strike K_1^b;
- long δ calls at strike K_2^b;
- long $(\beta - \delta)$ calls at strike K_3^b;
- long δ calls at strike K_4^b;

iii) the holdings of the X-denominated options on the Y are

- short $(\beta - \delta)$ puts at strike z_1;
- short δ calls at strike z_2.

This strategy would lead to the terminal payoff

$$
\begin{aligned}
g(x) &= \begin{cases} (\alpha-\lambda)(K_1-x)^+, & \text{if } x \le K_1; \\ 0, & \text{if } K_1 < x < K_2; \\ \lambda(x-K_2)^+, & \text{if } K_2 \le x < K_3; \\ \alpha(x-K_3)^+ + \lambda(K_3-K_2) & \text{if } K_3 \le x < K_4; \\ \lambda(x-K_4)^+ + \alpha(x-K_3)^+ + \lambda(K_3-K_2) & \text{if } K_4 \le x; \end{cases} \\
h(y) &= \begin{cases} (\beta-\delta)(K_1^b-y)^+, & \text{if } y \le K_1^b; \\ 0, & \text{if } K_1^b < y < K_2^b; \\ \delta(y-K_2^b)^+, & \text{if } K_2^b \le y < K_3^b; \\ \beta(y-K_3^b)^+ + \delta(K_3^b-K_2^b) & \text{if } K_3^b \le y < K_4^b; \\ \delta(y-K_4^b)^+ + \beta(y-K_3^b)^+ + \delta(K_3^b-K_2^b) & \text{if } K_4^b \le y; \end{cases} \\
f(z) &= \begin{cases} -(\beta-\delta)(z_1-z)^+, & \text{if } 0 < z \le z_1; \\ 0, & \text{if } z_1 < z < z_2; \\ -\delta(z-z_2)^+, & \text{if } z_2 \le z. \end{cases}
\end{aligned} \tag{48}
$$

From the program (12), there are three equalities:

$$
\begin{aligned}
&(\beta-\delta)z_1 = \lambda; \delta z_2 = \alpha - \lambda, \text{for } 0 < z_1 \le \tfrac{\alpha}{\beta} \le z_2; \\
&\lambda(K_3-K_2) + \delta(K_3^b - K_2^b) = \alpha K_3 + \beta K_3^b - \gamma (> 0),
\end{aligned} \tag{49}
$$

for $\lambda \in (0,\alpha)$ and $\delta \in (0,\beta)$. The equalities in (47) equivalently state that

$$
z_1 = \frac{K_3^b - K_1^b}{K_4 - K_2}, z_2 = \frac{K_4^b - K_2^b}{K_3 - K_1}. \tag{50}
$$

From these equalities, we can derive the following (in)equalities:

$$
\begin{aligned}
&\lambda(K_1-K_2) + \delta(K_3^b - K_4^b) = \alpha K_1 + \beta K_3^b - \gamma \le 0; \\
&\lambda(K_3-K_4) + \delta(K_1^b - K_2^b) = \alpha K_3 + \beta K_1^b - \gamma \le 0; \\
&\alpha K_4 + \beta K_2^b \ge \gamma; \alpha K_2 + \beta K_4^b \ge \gamma.
\end{aligned} \tag{51}
$$

These (in)equalities can establish the following results:

$$
g(x) + h(y) + xf(z)1_{z=y/x} - [\alpha x + \beta y - \gamma]^+ = \begin{cases} 0, \text{ if } 0 < x \le K_1 \text{ and } K_2^b \le y \le K_3^b; \\ 0, \text{ if } K_1 \le x \le K_2 \text{ and } K_1^b \le y \le K_2^b; \\ 0, \text{ if } K_2 \le x \le K_3 \text{ and } 0 < y \le K_1^b; \\ 0, \text{ if } K_2 \le x \le K_3 \text{ and } y \ge K_4^b; \\ 0, \text{ if } K_3 \le x \le K_4 \text{ and } K_3^b \le y \le K_4^b; \\ 0, \text{ if } x \ge K_4 \text{ and } K_2^b \le y \le K_3^b; \\ > 0, \text{otherwise}. \end{cases} \tag{52}
$$

Therefore, the strategy ϕ dominates the basket option in (4). As a result, the minimum cost of such a strategy would provide an upper price bound for this basket option:

$$
\begin{aligned}
\min_{\lambda,\delta,z_1,z_2,K_i,i=1,2,3,4} \int_{\mathbb{R}+} & [(\alpha-\lambda)(K_1-x)^+ + \lambda(x-K_2)^+ \\
& + (\alpha-\lambda)(x-K_3)^+ + \lambda(x-K_4)^+]\pi^X(x)dx \\
& + \int_{\mathbb{R}+} [(\beta-\delta)(K_1^b-y)^+ + \delta(y-K_2^b)^+ \\
& + (\beta-\delta)(y-K_3^b)^+ + \delta(y-K_4^b)^+]\pi^Y(y)dy \\
& + \int_{\mathbb{R}+} [-(\beta-\delta)(z_1-z)^+ + (-\delta)(z-z_2)^+]\pi^Z(z)dz
\end{aligned} \tag{53}
$$

where

$$
\begin{aligned}
&1)(\beta-\delta)z_1=\lambda;\delta z_2=\alpha-\lambda,\text{for }0\le z_1\le\tfrac{\alpha}{\beta}\le z_2\text{ and }\lambda\in(0,\alpha),\delta\in(0,\beta);\\
&2)\lambda(K_3-K_2)+\delta(K_3^b-K_2^b)=\alpha K_3+\beta K_3^b-\gamma;\\
&3)K_1^b=z_1K_2,K_2^b=z_2K_1,K_3^b=z_1K_4,K_4^b=z_2K_3;\\
&4)0\le\lambda<\alpha;0\le\delta<\beta;\\
&5)0\le K_1\le K_2\le K_3\le K_4.
\end{aligned}
$$

For the problem in (53), there exists a feasible solution by setting

$$
\begin{aligned}
&z_1\to 0;z_2\to\infty;K_1\to 0;K_4\to\infty;\lambda=0;\delta=0;\\
&K_2=K_3=K_a;z_2K_1=z_1K_4=K_b;z_1K_2\to 0;z_2K_3\to\infty;
\end{aligned}
\tag{54}
$$

where $\alpha K_a+\beta K_b=\gamma$. This solution leads to upper bounds in Proposition 3. The objective function in (53) is bounded above. Therefore, this program must have a solution which yields the least upper price bound on the basket option in (4).

2) We now build up a class of pricing functions $p(x,y)$ which support these dominating portfolios. Define two separated sets as follows:

$$
\begin{aligned}
A=&(0,K_1]\times[K_2^b,K_3^b]\cup[K_1,K_2]\times[K_1^b,K_2^b]\cup[K_2,K_3]\times(0,K_1^b]\\
&\cup[K_2,K_3]\times[K_4^b,\infty)\cup[K_3,K_4]\times[K_3^b,K_4^b]\cup[K_4,\infty)\times[K_2^b,K_3^b],\\
B=&\mathbb{R}_+^2\backslash A.
\end{aligned}
\tag{55}
$$

So the set A indicates the region where dominating strategies exactly replicate a basket option, and the set B is its complement.

When only dollar-denominated options are traded, a price function p is attained, associated with two strikes K_a^* and K_b^*, as shown in the left panel of Figure 1. Proposition 3 has established that

$$
p(x,y)=\begin{cases}\ge 0, & \text{if }(x,y)\in(0,K_a^*]\times(0,K_b^*];\\ \ge 0, & \text{if }(x,y)\in(K_a^*,\infty)\times(K_b^*,\infty);\\ 0, & \text{otherwise.}\end{cases}
$$

If the X-denominated options on the Y are traded, their quoted prices in markets may be consistent with the price function p so that valuation bounds are then not tightened. This scenario is linked to a feasible solution in (54). Otherwise, we consider the following construction.

I) Given the price function p, we first pick up two positive random numbers z_1 and z_2 so that $0<z_1<\frac{\alpha}{\beta}<z_2<\infty$. Four points from the X are chosen so that

$$0<K_1<K_2<K_a^*<K_3<K_4,$$

and four points from the Y are then determined

$$K_1^b=z_1K_2<K_2^b=z_2K_1<K_b^*<K_3^b=z_1K_4<K_4^b=z_2K_3.$$

Therefore, the joint density p is divided into 36 small partitions.

To construct a density shown in the right panel of Figure 1, the density of one node (x,y) in the set A is added by a small number $\epsilon>0$, and then the density of another node $(x,\tilde{y})$ or $(\tilde{x},y)$ in the set B is subtracted by ϵ. In this way, a new price function $\hat{p}$ that is consistent with the marginals π^X and π^Y is constructed from the density p.

II) Since the dollar-denominated options on the X and Y are properly priced under the new price function $\hat{p}$, the marginal condition implies that:

$$\mathbb{E}_{\hat{p}}[X]=\int_{(x,y)\in A}x\hat{p}(x,y)dxdy=\int_{(x,xz)\in A}x\hat{p}(x,xz)dxdz=x_0.$$

For $(x, xz) \in A$ and $\nu_z \sim U[0,1]$, define $\int_x x\hat{p}(x, xz)dx = x_0\nu_z$ so that $\int_z \int_x x\hat{p}(x, xz)dxdz = \int_z x_0\nu_z dz = x_0$.

To be consistent with the marginal π^Z, a new price function p^* is constructed from $\hat{p}$ so that $\int_x xp^*(x, xz)dx = x_0\pi^Z(z)$ for $(x, xz) \in A$. This requires that $\int_x x(\hat{p}(x, xz) - p^*(x, xz))dx = x_0(\nu_z - \pi^Z(z))$ and also $\int_z x_0(\nu_z - \pi^Z(z))dz = 0$. We pick up three random numbers z_a, z_b and z_c from the Z so that $z_a < z_b = \sqrt{z_a z_c} < z_c$ and

$$\nu_{z_a} - \pi^Z(z_a) + \nu_{z_b} - \pi^Z(z_b) + \nu_{z_c} - \pi^Z(z_c) = 0.$$

For two small positive numbers κ_1 and κ_2 ($\kappa_1(\nu_{z_b} - \pi^Z(z_b)) > \kappa_2(\nu_{z_c} - \pi^Z(z_c))$), three points from the X in the set A are chosen so that the adjustment of the density $\hat{p}$ at each node is reported as follows:

	z_a	z_b	z_c
x_a	0	-κ_1	+κ_1
x_b	+κ_1	$\kappa_2 - \kappa_1$	-κ_2
x_c	-κ_2	+κ_2	0

To be consistent with the density π^Z, all three points x_a, x_b and x_c ($x_b^2 = x_a x_c$) are attained by solving an equation system:

$$\begin{aligned}
x_a &= \frac{(\nu_{z_b} - \pi^Z(z_b))^2}{\kappa_1(\nu_{z_b} - \pi^Z(z_b)) - \kappa_2(\nu_{z_c} - \pi^Z(z_c))} x_0^2, \\
x_b &= \frac{(\nu_{z_b} - \pi^Z(z_b))(\nu_{z_c} - \pi^Z(z_c))}{\kappa_1(\nu_{z_b} - \pi^Z(z_b)) - \kappa_2(\nu_{z_c} - \pi^Z(z_c))} x_0^2, \\
x_c &= \frac{(\nu_{z_c} - \pi^Z(z_c))^2}{\kappa_1(\nu_{z_b} - \pi^Z(z_b)) - \kappa_2(\nu_{z_c} - \pi^Z(z_c))} x_0^2.
\end{aligned}$$

By repeating this process, we have $\int_z x_0(\nu_z - \pi^Z(z))dz = 0$. A new price function p^* is constructed from $\hat{p}$, associated with two adjustable variables κ_1 and κ_2. Hence, the X-denominated options on the Y are priced correctly under this price function.

Suppose there exists a group of parameters $(z_1^*, z_2^*, K_1^*, K_2^*, K_3^*, K_4^*)$ that support such price function p^*. Given these parameters, two separated sets A^* and B^* are determined so that $A^* \cup B^* = \mathbb{R}_+^2$. Then there must exist a candidate joint distribution:

$$p^*(x, y) = \begin{cases} \geq 0, & \text{if } (x, y) \in A^*; \\ = 0, & \text{if } (x, y) \in B^*, \end{cases} \tag{56}$$

so that $\int_{\mathbb{R}+} p^*(x, y)dy = \pi^X(x)$, $\int_{\mathbb{R}+} p^*(x, y)dx = \pi^Y(y)$ and $\int_{\mathbb{R}+} xp^*(x, xz)dx = \pi^Z(z)/x_0$.

The quantities of λ^* and δ^* are derived from the first condition in (53). Therefore, the dominating strategy provides the least upper bound on this basket option

$$\mathbb{E}_{p^*}[g^*(x) + h^*(y) + xf^*(z)1_{z=y/x} - (\alpha x + \beta y - \gamma)^+] = 0.$$

This condition ensures the existence of a candidate joint density specified in (56).

F Proof of Lemma 3

Proof First of all, define two new random variables by setting $\tilde{X} = \alpha X$ and $\tilde{Y} = \beta Y$ so that the payoff of a basket option is re-expressed as $[\tilde{X} + \tilde{Y} - \gamma]^+$ for $(\alpha, \beta, \gamma) \in \mathbb{R}_3^+$. According to Hobson, Laurence and Wang (2005b), partition $\mathbb{R}^+$ into $(2n+1)$ $(n \geq 1)$ finite intervals in the way:

$$0 = \tilde{K}_0^1 < \tilde{K}_1^1 < \cdots < \tilde{K}_{2n}^1 < \gamma < \tilde{K}_{2n+1}^1 = \infty,$$

so that $(0, \tilde{K}_1^1) \cup (\cup_{i=1}^{2n}[\tilde{K}_i^1, \tilde{K}_{i+1}^1)) \cup [\tilde{K}_{2n+1}^1, \infty) = (0, \infty)$. Let

$$\tilde{K}_i^2 = \gamma - \tilde{K}_{2n+1-i}^1.$$

Hence $\mathbb{R}_2^+$ may be expressed as the union of finite intervals:

$$\mathbb{R}_2^+ = \bigcup_{i,j=1}^{2n+1} R_{ij}, \text{for } R_{i,j} = \{(\tilde{x},\tilde{y}) \in \mathbb{R}_2^+ : \tilde{K}_{i-1}^1 \le \tilde{x} < \tilde{K}_i^1, \tilde{K}_{j-1}^2 \le \tilde{y} < \tilde{K}_j^2\}.$$

Now consider a portfolio which consists of two components:

1) the holdings of the dollar-denominated options on the X are expressed as follows:

$$f_X(\tilde{x}) = \tilde{x}^+ + \sum_{i=1}^{n} \{(\tilde{x} - \tilde{K}_{2i}^1)^+ - (\tilde{x} - \tilde{K}_{2i-1}^1)^+\}; \tag{57}$$

2) the holdings of the dollar-denominated options on the Y are expressed as follows:

$$f_Y(\tilde{y}) = \tilde{y}^+ + \sum_{j=1}^{n} \{(\tilde{y} - \tilde{K}_{2j}^2)^+ - (\tilde{y} - \tilde{K}_{2j-1}^2)^+\}, \tag{58}$$

associated with the amount of cash, $\omega = \sum_{l=1}^{n}(\tilde{K}_{2l}^2 - \tilde{K}_{2l-1}^2) - \gamma$.

At maturity, the payoff to this portfolio in each region R_{ij} would be equal to

$$\begin{aligned} f_X(\tilde{x}) + f_Y(\tilde{y}) + \omega &= f_X(\tilde{x}) - f_X(\gamma - \tilde{y}) \\ &= \tilde{x}^+ + \sum_{m=1}^{i\le n} ((\tilde{x} - \tilde{K}_{2m}^1)^+ - (\tilde{x} - \tilde{K}_{2m-1}^1)^+) \\ &\quad - \{(\gamma - \tilde{y})^+ + \sum_{k=1}^{j \le n} ((\tilde{K}_{2k-1}^2 - \tilde{y})^+ - (\tilde{K}_{2k}^2 - \tilde{y})^+)\}, \end{aligned} \tag{59}$$

due to $f_X(\gamma - \tilde{y}) + f_Y(\tilde{y}) + \omega = 0$.

Since the function f has the slope 0 or 1, the mean value theorem implies that

$$f_X(\tilde{x}) - f_X(\gamma - \tilde{y}) \le [\tilde{x} - (\gamma - \tilde{y})]^+ = [\alpha x - (\gamma - \beta y)]^+ = [\alpha x + \beta y - \gamma]^+. \tag{60}$$

In other words, this result may be re-expressed as

$$\begin{aligned} &[f_X(\tilde{x}) - f_X(\gamma - \tilde{y})] - [\tilde{x} + \tilde{y} - \gamma]^+ = \\ &\begin{cases} 0, & \text{if } \tilde{x} + \tilde{y} = \gamma; \\ 0, & \text{if } \tilde{x} + \tilde{y} > \gamma \text{ and } \tilde{x}, \gamma - \tilde{y} \in (\tilde{K}_{2i}^1, \tilde{K}_{2i+1}^1); \\ 0, & \text{if } \tilde{x} + \tilde{y} < \gamma \text{ and } \tilde{x}, \gamma - \tilde{y} \in (\tilde{K}_{2i-1}^1, \tilde{K}_{2i}^1); \\ < 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{61}$$

Therefore, the terminal payoff in (59) sub-replicates the basket option in the region R_{ij}, and thus it is dominated over $\mathbb{R}_2^+$. The initial cost provides a lower price bound on this option:

$$\begin{aligned} \underline{\mathcal{B}}(n) = &\int_{\mathbb{R}+} \alpha(x + \sum_{i=1}^{n}((x - \tilde{K}_{2i}^1/\alpha)^+ - (x - \tilde{K}_{2i-1}^1/\alpha)^+))\pi^X(x)dx \\ &- \int_{\mathbb{R}+} \beta((\frac{\gamma}{\beta} - y)^+ + \sum_{j=1}^{n}((\tilde{K}_{2j-1}^2/\beta - y)^+ - (\tilde{K}_{2j}^2/\beta - y)^+))\pi^Y(y)dy. \end{aligned} \tag{62}$$

Let $K_i^1 = \tilde{K}_i^1/\alpha$ and $K_j^2 = \tilde{K}_j^2/\beta$ $(1 \le i,j \le 2n+1)$. To see that the bound $\underline{\mathcal{B}}(n)$ is the non-increasing function of partition number n. Recall the terminal payoff to the hedging portfolio:

$$\begin{aligned} &f_X(\alpha x) + f_Y(\beta y) + \omega = (-\alpha)(K_1^1 - x)^+ + \alpha(x - K_2^1)^+ + (-\beta)(K_{2n-1}^2 - y)^+ \\ &+ \beta(y - K_{2n}^2)^+ + (\alpha K_1^1 + \beta K_{2n}^2 - \gamma) + \alpha\sum_{i=2}^{n}((x - K_{2i}^1)^+ - (x - K_{2i-1}^1)^+) \\ &+ \beta\sum_{j=1}^{n-1}((y - K_{2j}^2)^+ - (y - K_{2j-1}^2)^+) + \beta(\sum_{l=1}^{n-1}(K_{2l}^2 - K_{2l-1}^2)) \\ &= (-\alpha)(K_1^1 - x)^+ + \alpha(x - K_2^1)^+ + (-\beta)(K_{2n-1}^2 - y)^+ + \beta(y - K_{2n}^2)^+ \le [\alpha x + \beta y - \gamma]^+, \end{aligned} \tag{63}$$

where $\alpha K_1^1 + \beta K_{2n}^2 = \alpha K_2^1 + \beta K_{2n-1}^2 = \gamma$, because for each $x \in [K_{2i-1}^1, K_{2i}^1)$, there exists a $y = \frac{\gamma - \alpha x}{\beta} \in [K_{2n+1-2i}^2, K_{2n+2-2i}^2)$ so that

$$\alpha \sum_{i=2}^{n} ((x - K_{2i}^1)^+ - (x - K_{2i-1}^1)^+) + \beta \sum_{j=1}^{n-1} ((y - K_{2j}^2)^+ - (y - K_{2j-1}^2)^+) + \beta(\sum_{l=1}^{n-1} (K_{2l}^2 - K_{2l-1}^2))$$
$$= 0 \tag{64}$$

Now consider a strategy that involves selling α puts and buying α calls on the X with strikes K_1^1 and K_2^1, and selling β puts and buying β calls on the Y with strikes K_{2n-1}^2 and K_{2n}^1. This strategy sub-replicates the payoff $[\alpha x + \beta y - \gamma]^+$. In sum, the bound $\underline{\mathcal{B}}(n)$ is the decreasing function of partition number n.

G Proof of Proposition 6

Proof Given a triplet $(\alpha, \beta, \gamma) \in \mathbb{R}_3^+$, Lemma 3 implies that sub-replicating strategies for a basket option consist of two components:

i) the holdings of the dollar-denominated options on the X involve short α puts at strike K_a^1 and long α calls at strike K_a^2;

ii) the holdings of the dollar-denominated options on the Y involve short β puts at strike K_b^1 and long β calls at strike K_b^2,

where $\alpha K_a^1 + \beta K_b^2 = \alpha K_a^2 + \beta K_b^1 = \gamma$. It is easy to verify that the terminal payoffs generated by these strategies would sub-replicate the basket option in the way:

$$g(x) + h(y) - [\alpha x + \beta y - \gamma]^+ = \begin{cases} 0, \text{ if } (x, y) \in (0, K_a^1] \times [K_b^2, \infty) \text{ and } \alpha x + \beta y \geq \gamma; \\ 0, \text{ if } (x, y) \in [K_a^1, K_a^2] \times [K_b^1, K_b^2] \text{ and } \alpha x + \beta y \leq \gamma; \\ 0, \text{ if } (x, y) \in [K_a^2, \infty) \times (0, K_b^1] \text{ and } \alpha x + \beta y \geq \gamma; \\ < 0, \text{otherwise}. \end{cases} \tag{65}$$

Given a triplet $(\alpha, \beta, \gamma) \in \mathbb{R}_3^-$, we consider trading strategies as follows

i) the holdings of the dollar-denominated options on the X involve short α puts at strike K_a^1, long α calls at strike K_a^2, and long α calls and short α puts at strike K_X;

ii) the holdings of the dollar-denominated options on the Y involve short β puts at strike K_b^1, long β calls at strike K_b^2, and long β calls and short β puts at strike K_Y,

where $\alpha K_a^1 + \beta K_b^2 = \alpha K_a^2 + \beta K_b^1 = \alpha K_X + \beta K_Y = \gamma$. It is easy to verify that the terminal payoffs generated by these strategies would sub-replicate the basket option as shown in (65). Therefore, the valuation bounds on this option can be represented in (17) and (18). The solution sets of these programs are not empty, provided that there always exist strikes for K_a^1, K_a^2, K_b^1, K_b^2, K_X and K_Y. Also, these programs are bounded from above and thus must have solutions.

To see that there exists a price function p that supports the bound in (17) or (18), consider the construction of a counter-monotonic process (X, Y) so that the variable Y is a non-increasing function of the variable X. More specifically, there exists a real number vector $(\bar{x}, \bar{y}) \in \mathbb{R}_2^+$ so that

$$prob(X \leq \bar{x}) = \nu; prob(Y \leq \bar{y}) = 1 - \nu, \text{ for } \nu \sim U[0, 1]. \tag{66}$$

Then a counter-monotonic process (X, Y) is constructed through the inverse function, and the variables X and Y have the desired marginal densities. The joint density of (X, Y) is specified in (19).

H Proof of Proposition 7

Proof 1) Given a triplet $(\alpha, \beta, \gamma) \in \mathbb{R}_3^+$, we shall set up a class of dominated hedging portfolios for the option **b**. First of all, pick up four points from the X and four points from the Y:

$$0 < K_1 < K_2 < K_3 < K_4; 0 < K_1^Y < K_2^Y < K_3^Y < K_4^Y, \tag{67}$$

so that there exists two variables z_1 and z_2 $(0 < z_1 \leq z_2 < \infty)$

$$\text{i) } z_1 = \frac{K_1^Y}{K_3} = \frac{K_2^Y}{K_4}, z_2 = \frac{K_3^Y}{K_1} = \frac{K_4^Y}{K_2}, \tag{68}$$

and

$$\text{ii) } \alpha K_1 + \beta K_3^Y = \gamma; \alpha K_2 + \beta K_2^Y = \gamma; \alpha K_3 + \beta K_1^Y = \gamma. \tag{69}$$

To sub-replicate the payoff of a basket option, consider a trading strategy:

i) the holdings of the dollar-denominated options on the X consist of long λ calls at strike K_3 and short $(\lambda - \alpha)$ call at strike K_4;

ii) the holdings of the dollar-denominated options on the Y consist of long δ calls at strike K_3^Y and short $(\delta - \beta)$ call at strike K_3^Y;

iii) the holdings of the X-denominated options on the Y consist of short β puts at strike z_1, and short $(\delta - \beta)$ calls at strike z_2,

for $\lambda \geq \alpha$ and $\delta \geq \beta$.

Following the conditions in (20),

$$\begin{aligned} &\beta z_1 = \lambda - \alpha; (\delta - \beta) z_2 = \alpha, \text{for } 0 < z_1 \leq z_2 < \infty; \\ &\lambda(K_4 - K_3) + \delta(K_4^Y - K_3^Y) = \alpha K_4 + \beta K_4^Y - \gamma, \end{aligned} \tag{70}$$

we can identify the following equalities, associated with (68) and (69):

$$\begin{aligned} &\lambda K_3 = \delta K_3^Y = \gamma; \\ &(\lambda - \alpha) K_4 + (\delta - \beta) K_4^Y = \gamma. \end{aligned} \tag{71}$$

With these equalities, the strategy identified above sub-replicates this basket option:

$$\begin{aligned} &g(x) + h(y) + x f(z) 1_{z=y/x} - [\alpha x + \beta y - \gamma]^+ = \\ &\begin{cases} 0, & \text{if } 0 < x \leq K_3 \text{ and } 0 < y \leq K_3^Y \text{ for } \frac{y}{x} \in [z_1, z_2] \text{ and } \alpha x + \beta y \leq \gamma; \\ 0, & \text{if } 0 < x \leq K_2 \text{ and } K_3^Y < y \leq K_4^Y \text{ for } \frac{y}{x} \geq z_2 \text{ and } \alpha x + \beta y \geq \gamma; \\ 0, & \text{if } K_3 < x \leq K_4 \text{ and } 0 < y \leq K_2^Y \text{ for } \frac{y}{x} \leq z_1 \text{ and } \alpha x + \beta y \geq \gamma; \\ 0, & \text{if } x > K_4 \text{ and } y > K_4^Y \text{ for } \frac{y}{x} \in [z_1, z_2]; \\ < 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{72}$$

Note that since there always exit the strikes K_i and K_i^Y $(i = 1, 2, 3)$ from the construction above, these strikes then determine the strikes z_1 and z_2 from the condition (68). As a result, the strikes K_4 and K_4^Y are also obtained from that condition. Hence, all the relevant strikes are determined once the strikes K_i and K_i^Y $(i = 1, 2, 3)$ are chosen. This further implies that a dominated trading strategy identified above always exists. Moreover, the strategies identified in (17) can be viewed as the particular cases with the zero holdings of the X-denominated options on the Y for $z_1 \to 0$ and $z_2 \to \infty$.

Therefore, the lower valuation bounds on the basket option are attained as follows:

$$\begin{aligned} \max_{\lambda, \delta, z_1, z_2, K_i, K_i^y, i=1,2,3,4} &\int_{\mathbb{R}^+} (\lambda(x - K_3)^+ + (\alpha - \lambda)(x - K_4)^+) \pi^X(x) dx \\ &+ \int_{\mathbb{R}^+} (\delta(y - K_3^Y)^+ + (\beta - \delta)(y - K_4^Y)^+) \pi^Y(y) dy \\ &+ \int_{\mathbb{R}^+} ((-\beta)(z_1 - z)^+ + (\beta - \delta)(z - z_2)^+) \pi^Z(z) dz, \end{aligned} \tag{73}$$

s.t.

$$
\begin{aligned}
&1)\beta z_1 = \lambda - \alpha; (\delta - \beta) z_2 = \alpha, \text{for } 0 \leq z_1 \leq z_2;\\
&2)\lambda(K_4 - K_3) + \delta(K_4^Y - K_3^Y) = \alpha K_4 + \beta K_4^Y - \gamma;\\
&3)\alpha K_1 + \beta K_3^Y = \gamma; \alpha K_3 + \beta K_1^Y = \gamma;\\
&4)\lambda \geq \alpha; \delta \geq \beta; 0 \leq K_1 \leq K_2 \leq K_3 \leq K_4.
\end{aligned}
$$

According to the analysis above, this problem always has a feasible solution and it is bounded above. Thus it must have a solution. Since the strategies identified in (17) are the special cases of $f(z) \equiv 0$, it is equivalent to the case of $K_a^1 \to 0$ and $K_b^1 \to 0$ and thus the greatest lower bound derived from this problem should not be cheaper than the bound attained in (17).

The rest of the proof is about the construction of the pricing function that supports the greatest lower bound. When only dollar-denominated options are traded, a price function p is attained and the strikes $\hat{K}_a^1$, $\hat{K}_a^2$, $\hat{K}_b^1$ and $\hat{K}_b^2$ are attained, as shown in the left panel of Figure 2. Under this pricing function, the dollar-denominated options on the X and Y are priced correctly. However, this function may be inconsistent with the prices of the X-denominated options on the Y.

Given the pricing function p, we firstly pick up two positive numbers $z_1 = \frac{\hat{K}_b^1}{\hat{K}_a^2}$ and $z_2 = \frac{\hat{K}_b^2}{\hat{K}_a^1}$. Then one can draw the lines through these numbers to divide the space into three regions. Furthermore, another four points are chosen in the way:

$$
\alpha K_x^2 + \beta K_y^2 = \gamma; K_x^4 = \frac{K_y^2}{z_1}; K_y^4 = z_2 K_x^2,
$$

where $K_x^2 \in (\hat{K}_a^1, \hat{K}_a^2)$. Following the first procedure in the proof of Proposition 4, one can construct a pricing function $\hat{p}$ that is consistent with dollar-denominated options. To be consistent with cross-rate options, one follow the second procedure to construct a density function p^* by adjusting K_x^2 and K_y^2.

Therefore, one can construct a pricing function p^* by choosing a group of parameters $(z_1, z_2, \hat{K}_a^1, \hat{K}_x^2, \hat{K}_a^2, \hat{K}_x^4)$:

$$
p^*(x,y) = \begin{cases}
\geq 0, & \text{if } (x,y) \in (0, \hat{K}_a^2] \times (0, z_2 \hat{K}_a^1] \text{ such that } \frac{y}{x} \in [\hat{z}_1, \hat{z}_2] \text{ and } \alpha x + \beta y \leq \gamma;\\
\geq 0, & \text{if } (x,y) \in (0, \hat{K}_x^2] \times [z_2 \hat{K}_a^1, z_2 \hat{K}_x^2] \text{ such that } \alpha x + \beta y \geq \gamma\ ;\\
\geq 0, & \text{if } (x,y) \in [\hat{K}_a^2, \hat{K}_x^4] \times (0, z_1 \hat{K}_x^4] \text{ such that } \alpha x + \beta y \geq \gamma\ ;\\
\geq 0, & \text{if } (x,y) \in [\hat{K}_x^4, \infty) \times [z_2 \hat{K}_x^2, \infty) \text{ such that } \frac{y}{x} \in [\hat{z}_1, \hat{z}_2];\\
0, & \text{otherwise.}
\end{cases}
$$

Also, one can construct a dominated trading strategy, while the quantities of λ and δ are determined in the first condition of the problem (73). With this pricing function, there is the following relation:

$$
\mathbb{E}_{p^*}[g^*(x) + h^*(y) + x f^*(z) 1_{z=y/x} - [\alpha x + \beta y - \gamma]^+] = 0.
$$

2) For a triplet $(\alpha, \beta, \gamma) \in \mathbb{R}_3^-$, the analysis can be completed in the similar way. First of all, four points from the X and four points from the Y:

$$
0 < K_1 < K_2 < K_3 < K_4; 0 < K_1^Y < K_2^Y < K_3^Y < K_4^Y, \tag{74}
$$

so that

$$
\begin{aligned}
&K_1^Y = z_1 K_3, K_2^Y = z_1 K_4, K_3^Y = z_2 K_1, K_4^Y = z_2 K_2, \text{for } 0 < z_1 \leq z_2 < \infty\ ,\\
&\alpha K_1 + \beta K_3^Y = \gamma; \alpha K_2 + \beta K_2^Y = \gamma; \alpha K_3 + \beta K_1^Y = \gamma.
\end{aligned} \tag{75}
$$

Following the conditions in (20)

$$
\begin{aligned}
&\beta z_1 = -\lambda; \delta z_2 = -\alpha, \text{for } 0 < z_1 \leq z_2 < \infty;\\
&\lambda(K_3 - K_4) + \delta(K_3^Y - K_4^y) = \alpha K_3 + \beta K_3^Y - \gamma,
\end{aligned} \tag{76}
$$

we can identify the following equalities, associated with (75) and (69):

$$\begin{aligned}&(-\alpha)(K_3 - K_2) + \lambda(K_3 - K_4) = 0;\\&(-\beta)(K_3^Y - K_2^Y) + \delta(K_3^Y - K_4^Y) = 0.\end{aligned} \tag{77}$$

With these equalities, the the strategy identified in (21) sub-replicates the basket option. Following the same construction in Proposition 4, there exists a joint density that supports the greatest lower valuation bound on this option.