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Issue 5 – August 2014

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Modelling Consumer Disaffection with Local Pricing

Simon Rudkin*

ABSTRACT

Many firms set identical prices in differentiated geographic markets, despite having near costless access to data and price varying technology, contradicts the established view that price discrimination would be profit improving. Motivated by UK supermarket competition analysis we present a new model, in a Salop (1979) setting, in which consumers feel disaffection towards firms who charge them more for a product they sell cheaper elsewhere. National pricing becomes a Nash equilibrium for multi market retailers. Welfare is improved compared to the standard Salop results but long run entry remains excessive.

Key Words: Firm Behaviour, Market Structure, Oligopoly, Pricing.

Acknowledgements: The author would like participants in the seminars at the University of Manchester and Newcastle University Business School for their constructive comments on earlier drafts of this work and Professor Paul Madden for his helpful suggestions.

JEL Classification: L10 L11 D43

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1. INTRODUCTION

Firms seek to gain maximum profit at the aggregate level, and hence would be expected to do so in each market they are active in. Indeed, Robinson (1969) concluded that a monopolist facing two consumers with different demand elasticities can "increase his gains by selling to them at a different price if it is possible to do so". This intuitive conclusion that third degree price discrimination, the practice of setting different prices in separate markets, is optimal is supported by a wide literature[1], real world examples[2], and simple exercises in the Hotelling (1929) and Salop (1979) frameworks[3]. However, Smith (2004), Competition Commission (2003) and Dobson and Waterson (2008)[4] all point to many firms who choose not to flex prices, particularly in the UK supermarket industry[5]. Here we seek to explore why, despite there being no notable barriers to doing so[6], we might not see different prices for each consumer group. A notion of disaffection against firms who charge more for a product that is available cheaper from one of their stores elsewhere is developed, based on suggestions by Levy et al (2004), the supermarkets in evidence to the Competition Commission (2003) and by Dobson and Waterson (2008)[7].

Introducing competition to the market place creates a new dynamic and necessitates further terminology. A market is strong (weak) for a firm if it would wish to increase (reduce) price therein. Competitors may view the same market as their strong market, symmetric ranking, or may feel that

[1] For a comprehensive analysis of the literature on monopoly and price discrimination see Kwon (2006)
[2] Ning and Haining (2004) find third degree price discrimination by UK Petrol stations, Villas-Boas (2009) find it in German retail, and Hviid and Price (2012) show this is true in the UK retail energy sector.
[3] Ikeda and Toshimitsu (2010) study monopoly in a market with and without an endogenous quality measure. This reaffirms and extends the conclusion that third degree price discrimination is optimal for a monopolist who is able to separate their markets.
[4] Dobson and Waterson (2008) show price flexing by Tesco fell from 8.5% in 1999 to zero in 2003, Sainsbury also showed a reduction to 0 for the same period. A full table of changes is presented on page 7 of the paper.
[5] For our purposes we define supermarkets as Asda owned by Wal-Mart inc., Tesco plc, W M Morrison plc and J Sainsbury plc, these are the four major national chains in 2008. Further examples are provided in Dobson and Waterson (2005).
[6] Early work assumed costs of charging different prices would potentially outweigh the extra profits gained, however modern technology, particularly computerised check out systems, have greatly reduced these costs.
[7] The authors observe that shoppers develop "adverse sentiment over geographic price discrimination".

they have different strong markets, asymmetric ranking. Holmes (1989) early work on symmetric rankings finds discrimination may move firm's prices in the least profitable direction if cross price elasticity is large, and industry level price elasticity small, in the weak relative to the strong market[8]. This offers potential for uniform pricing to be a profit improving strategy. Armstrong and Vickers (2001) explore the Holmes result using a pair of firms operative on two Hotelling (1929) linear cities, the strong market being that with the highest transport cost, as it is in this paper. Irrespective of the proportion of the population residing in the strong market, it is always found that discrimination is the most profitable strategy. Our results show the degree of competition and relative transport costs are indeed what shapes the range over which uniform pricing is practiced in a symmetrical rank setting. However, alone, they will not provide uniform pricing as a Nash equilibrium as per Armstrong and Vickers.

Corts (1998) drops the assumption of symmetric ranking, motivated by an example of competition between traditional high street retailers and discount outlets[9]. Uniform pricing can be most profitable, but firms retain an incentive to set local prices given the price of the other store. The game is like a prisoners' dilemma, incentives to deviate denying the most profitable equilibrium. Consequently a first stage is posited at which firms may commit to uniform pricing prior to price competition. In these games national pricing can prevent all out competition between the firms[10], but unless the commitment is binding it will still not be a sub game perfect Nash equilibrium. Both Schulz (1999) and Dobson and Waterson (2005) find similar uses for a binding commitment[11]. In industries where consumers buy many different products, or comparing price is costly to consumers, such commitments would be reduced in credibility. Corts concludes by stating that, absent of any

[8] Ghost Datsidar (2006) also studies the symmetrical ranking of markets by duopolists with similar results.
[9] Corts (1998) example is illustrated by a New York Times article on the competition between the traditional department stores and the new discounters that have opened nearby.
[10] They must lower their price in their own strong market in response to the other firm raising its price therein.
[11] This commitment could be achieved through the issuing of catalogues or price comparison websites Dobson and Waterson (2008) cite the example of www.tesco.com/pricecheck

obvious demand or cost incentive to nationally price, a necessary condition for uniform pricing to be most profitable is that at least one firm values each market as being its strongest. Here market ranking is symmetric so Corts' conditions cannot be met and something more is needed.

That individuals are willing to incur personal cost to punish those who they feel treat them unfairly is shown in Fehr and Fischbacher (2002). In this paper we rely on consumers incurring the further disutility of "disaffection" when they are charged too much for a product that is available cheaper elsewhere from the same firm. By considering that supermarket chains aggregate their profits at a national level, and have a national identity, it is reasonable for consumers to assume pricing policy would be set in a similar way. Indeed the marketing language of UK supermarket chains supports this view through talk of "everyday low prices" and "value", both of which could be readily compromised by setting different prices in different markets. While a necessary assumption of the model is that consumers can not travel to other markets, or trade across markets, it is not unreasonable to think that they might care about prices elsewhere and or be able to gather information about them[12]. Whether motivated by local identity or pure self interest it is quite inevitable that anyone would feel disutility on learning they are over paying for something that is available cheaper elsewhere. Store facilities may differ across markets, indeed these are often used in lieu of price in local competition (Dobson and Waterson, 2008) but it does not follow that one set of facilities would necessarily be associated with one market type, nor that all consumers would factor all facilities into their opinions on pricing.

Motivated by practices such as supermarket membership discounts and peak pricing Wu et al (2012) review consumer attitudes to discrimination within a particular store, noting discriminators engender negative feeling amongst disadvantaged consumers facing discrimination on their social or physical attributes. From this it could be inferred that similar negativity might be experienced when

[12] This could be done by talking with friends in other areas of the country, reading news reports about pricing levels or similar.

the physical attribute in question is the market of residence and the firm operates in multiple markets, the motivation for this paper. Hence we generate a measure based on price differential which recognises consumers understanding of their local environment but acknowledges their annoyance at paying more and being lied to by the stores marketing.

Rotemberg (2005) studies a single firm's decision to change price when its costs increase in the face of shoppers who will not buy if they feel they are being unfairly treated. This slower rate of adjustment is studied in many applications. Hofstetter and Tovar (2010) show how consumer reference prices can guide firms' actions in the Colombian retail gasoline market[13]. Firms wishing to charge more than the government reference price can only adjust prices upwards slowly because of demand pressure. Here no consideration is given to multiple time periods, but the reference price could easily be seen as the price in other markets.

First the model is outlined for two market types containing only chain stores. Having identified conditions under which uniform pricing would be optimal we show freedom of entry and exit to the market will create more chains than is socially optimal[14] . The presence of disaffection reduces the magnitude of the effect.

2. THE MODEL

There are n firms and each has a branch on both of two markets, A and B. Each market is represented by a Salop (1979) circle of perimeter 1 and has a uniform, mass 1 distribution of consumers resident around its circumference. These shoppers must pay $t^X d$, $X = A, B$ in order to access a store located arc distance d from their residence in market X . Consumers may not travel,

[13] Similar results are found in studies by Peltzman (2000) and Meyer and von Cramon-Taubadel (2004).
[14] Consistent with the standard over branching result found in Salop (1979) style models.

or trade, between markets. Without loss of generality we label the markets such that $t^A \geq t^B$. Each consumer buys just one unit from one store[15]. A shopper who finds that a store is charging a higher price in their market than in the other market feels a disaffection that is proportional to the extent of the overcharging. We assume a sufficiently high valuation of the good to ensure that all shoppers make a purchase. Total cost for a shopper in market X from making a purchase at store i is thus:

$$p_i^X + t^X d + max(\mu(p_i^X - p_i^Y), 0) \tag{1}$$

Here p_i^X is the price of firm i in market X and p_i^Y is the price charged by the same firm in the other market. $\mu(> 0)$ captures the level of disaffection about prices being higher elsewhere[16]. Our linear specification for disaffection simplifies the exposition but similar results will appear wherever the size of price differential matters.

All stores locate symmetrically around the perimeter. In the short run firm i, $i = 1,...,n$ selects a pair of prices p_i^A, p_i^B in order to maximise profits π_i. Using (1) we can calculate the short run demand for firm i, D_i^X, finding the location $\bar{d}_{\pm}$ of the consumer indifferent between purchase at i or $i \pm 1$ and then summing. Because of the dependence on price ranking derivation of the demand expressions follows in section 3. Marginal costs are identical for all firms and normalised to zero, such that:

$$\pi_i = D_i^A p_i^A + D_i^B p_i^B \tag{2}$$

[15] This could also be seen as a basket of goods which are sold at all stores, but it is assumed here to simplify the exposition. Extending the model to analyse multiple product purchase is left for a future extension.

[16] In certain cases μ may be negative, for example if people preferred firms who offered discounts to those less well off than themselves. In the food shopping application there is no reason to assume that people would want their store to offer lower prices to someone in another geographic region.

Firm i may thus choose to locally price, setting $p_i^A \neq p_i^B$, or uniform price, setting $p_i^A = p_i^B = p^U$. In the long run we assume that chains are free to enter or exit the market but that any new firm must open a branch in each market[17]. Entry takes place until it is no longer profitable, that is $\Pi_i = \pi_i - f = 0$, where f is the fixed cost of entry.

3. SHORT RUN

In the short run n is taken as given. We look for symmetric equilibrium in the sense that all stores in market X set the same price, so $p_i^X = p^X$, $i = 1,...,n$. The precise nature of demand will depend on the market in which the prices charged by firm i are the highest and the market in which those charged by other firms are the highest.

Calculation of demands follows the same process in every case. For example, for the top line of (3), regime 1, we find $\bar{d}_+^A$ in market A who is indifferent between purchasing from firm i and firm $i+1$. Hence from (1), we find that $p_i^A + t^A \bar{d}_A^+ + \mu(p_i^A - p_i^B) = p^A + t^A\left(\frac{1}{n} - \bar{d}_+^A\right) + \mu(p^A - p^B)$. We then rearrange for $\bar{d}_+^A$ and double[18] doubling to reflect the consumer indifferent between firm $i-1$ and firm i.

$$\left.\begin{aligned} D_i^A &= \frac{1}{t^A}\left[(1+\mu)(p^A - p_i^A) + \mu(p_i^B - p^B) + \frac{t^A}{n}\right] \\ D_i^B &= \frac{1}{t^B}\left[p^B - p_i^B + \frac{t^B}{n}\right] \end{aligned}\right\} \begin{matrix} If\ p_i^A \geq p_i^B\ and\ p^A \geq p^B \\ (Regime\ 1) \end{matrix} \qquad (3)$$

$$\left.\begin{aligned} D_i^A &= \frac{1}{t^A}\left[p^A - p_i^A + \frac{t^A}{n}\right] \\ D_i^B &= \frac{1}{t^B}\left[(1+\mu)(p^B - p_i^B) + \mu(p_i^A - p^A) + \frac{t^B}{n}\right] \end{aligned}\right\} \begin{matrix} If\ p_i^A \leq p_i^B\ and\ p^A \leq p^B \\ (Regime\ 2) \end{matrix} \qquad (4)$$

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[17] Economies of scale might mean operation in just one market could not be profitable, for example.
[18] Firm $i-1$ is setting identical prices to firm i by assumption.

$$\left.\begin{aligned} D_i^A &= \frac{1}{t^A}\left[p^A - (1+\mu)p_i^A + \mu p_i^B + \frac{t^A}{n}\right] \\ D_i^B &= \frac{1}{t^B}\left[(1+\mu)p^B - p_i^B - \mu p^A + \frac{t^B}{n}\right] \end{aligned}\right\} \begin{array}{c} If\ p_i^A \geq p_i^B\ and\ p^A \leq p^B \\ (Regime\ 3) \end{array} \tag{5}$$

$$\left.\begin{aligned} D_i^A &= \frac{1}{t^A}\left[(1+\mu)p^A - p_i^A + \mu p^B + \frac{t^A}{n}\right] \\ D_i^B &= \frac{1}{t^B}\left[p^B - (1+\mu)p_i^B + \mu p_i^A + \frac{t^B}{n}\right] \end{aligned}\right\} \begin{array}{c} If\ p_i^A \leq p_i^B\ and\ p^A \geq p^B \\ (Regime\ 4) \end{array} \tag{6}$$

Where t^X is high shoppers near a store find it prohibitively expensive to switch to an alternative outlet giving their local outlet increased market power. These demand functions are decreasing in transport costs as would be expected. Hence as the transport costs fall the market becomes more competitive and the effects of the relative prices become more pronounced.

Lemma 1: **Within each pricing regime the payoff function for firm i is strictly concave in p_i^A, p_i^B if $\frac{t^A}{t^B} \in \left(\frac{\mu^2}{4(1+\mu)}, \frac{4(1+\mu)}{\mu^2}\right)$.**

This range is non empty if $\mu \leq 2 + 2\sqrt{2}$.

Proof: Substituting for demand in (2) in regimes 1 and 3 the corresponding hessian matrices for π_i are thus $\begin{bmatrix} -\frac{2(1+\mu)}{t^A} & \frac{\mu}{t^A} \\ \frac{\mu}{t^A} & -\frac{2}{t^B} \end{bmatrix}$, while in 2 and 4 the corresponding hessian is $\begin{bmatrix} -\frac{2}{t^A} & \frac{\mu}{t^B} \\ \frac{\mu}{t^B} & -\frac{2(1+\mu)}{t^B} \end{bmatrix}$.

These hessians are negative definite, profits are strictly concave, whenever $\frac{t^A}{t^B} > \frac{\mu^2}{4(1+\mu)}$ and $\frac{t^A}{t^B} < \frac{4(1+\mu)}{\mu^2}$ respectively. The range $\frac{t^A}{t^B} \in \left(\frac{\mu^2}{4(1+\mu)}, \frac{4(1+\mu)}{\mu^2}\right)$ is non empty whenever $\mu \leq 2 + 2\sqrt{2}$ by rearrangement.

Proposition 1: If the parameters satisfy the conditions in Lemma 1 then

there exists a unique symmetric Nash Equilibrium p^A, p^B with:

Uniform Pricing: $$p^A = p^B = p^U = \frac{2t^A t^B}{n(t^A + t^B)} \qquad \textbf{if} \qquad \frac{t^A}{t^B} \in [1, 1+2\mu]$$

Local Pricing: $$p^A = \frac{t^A}{n(1+\mu)} > p^B = \frac{(2\mu+1)t^B}{n(1+\mu)} \qquad \textbf{if} \qquad \frac{t^A}{t^B} > 1+2\mu$$

Figure 1 illustrates the parameter regions that form the basis of the proof. Regions C,D,E and F on the plot do not satisfy the concavity and/or transport cost requirements. Region G corresponds to uniform pricing, region H to local pricing.

Figure 1 here

The parameter regions illustrate that where the differential between market transport costs is large the incentive to price discriminate outweighs the disaffection effect. However as μ increases region H becomes smaller with stronger motivation for firms to uniform price.

In order to prove Proposition 1we will show that Firm i will not deviate from the proposed equilibrium to set any other:

(i) $p_i^A \geq p_i^B$ when the parameters are in region G

(ii) $p_i^A < p_i^B$ when the parameters are in region G

(iii) $p_i^A \geq p_i^B$ when the parameters are in region H

(iv) $p_i^A < p_i^B$ when the parameters are in region H.

First, for (i). Demands for firm i are given by (3) with $p^A = p^B = p^U$. Consider the maximisation problem for firm i, $\underset{p_i^A, p_i^B}{Max}\,\pi_i$ subject to $p_i^A \geq p_i^B$ whose Lagrangean is $L_i = \pi_i + \lambda\left(p_i^A - p_i^B\right)$. Both objective and constraint functions are concave in prices and so the Kuhn Tucker conditions are necessary and sufficient for a solution to this problem. Assuming that the constraint is binding and firm i sets $p_i^A = p_i^B = p_i$ say. $\frac{\partial L_i}{\partial p_i^A} = 0$ and $\frac{\partial L_i}{\partial p_i^B} = 0$ give us

$\lambda = \frac{\mu p_i + 2}{t^A} + \frac{p^U}{t^B} - \frac{1}{n} = \frac{\mu p_i + p^U}{t^A} - \frac{2p_i}{t^B} - \frac{1}{n} \Rightarrow p_i = \frac{2t^A t^B}{n\left(t^A + t^B\right)} = p^U$ with $\lambda \geq 0$, such that the constraint does indeed bind, whenever $\frac{t^A}{t^B} \leq 1 + 2\mu$.

For the second case, case (ii), demands for firm i are now given by (4) with $p^A = p^B = p^U$. Consider the maximisation problem for firm i, $\underset{p_i^A, p_i^B}{Max}\,\pi_i$ subject to $p_i^A \leq p_i^B$ whose Lagrangean is $L_i = \pi_i + \lambda\left(p_i^B - p_i^A\right)$. Both objective and constraint functions are concave in prices and so the Kuhn Tucker conditions are necessary and sufficient for a solution to this problem. Assuming that the constraint is binding and firm i sets $p_i^A = p_i^B = p_i$ say. $\frac{\partial L_i}{\partial p_i^A} = 0$ and $\frac{\partial L_i}{\partial p_i^B} = 0$ give us

$\lambda = \frac{(\mu - 2)p_i}{t^A} + \frac{p^U}{t^B} + \frac{1}{n} = \frac{(\mu + 2)p_i}{t^B} - \frac{p^U}{t^B} - \frac{1}{n} \Rightarrow p_i = \frac{2t^A t^B}{n\left(t^A + t^B\right)} = p^U$ with $\lambda \geq 0$, such that the constraint does indeed bind, whenever $\frac{t^A}{t^B} > \frac{1}{2\mu + 1}$. Since $\mu > 0$ and $t^A \geq t^B$ this is true throughout.

In case (iii) demands are given by (3). Firm Consider the maximisation problem for firm i, $\underset{p_i^A, p_i^B}{Max}\, \pi_i$ subject to $p_i^A \geq p_i^B$ whose Lagrangean is $L_i = \pi_i + \lambda\left(p_i^A - p_i^B\right)$. Both objective and constraint functions are concave in prices and so the Kuhn Tucker conditions are necessary and sufficient for a solution to this problem. The local solution $p_i^A > p_i^B$ has $\lambda = 0$, with $\frac{\partial L_i}{\partial p_i^A} = 0 \Rightarrow p_i^A = \frac{t^A}{n(1+\mu)} = p^A$ and $\frac{\partial L_i}{\partial p_i^B} = 0 \Rightarrow p_i^B = \frac{(1+2\mu)t^B}{n(1+\mu)} = p^B$. This satisfies $p_i^A > p_i^B$ whenever $\frac{t^A}{t^B} > 1 + 2\mu$.

Finally we need to prove that (iv) is true. Demands are given by (6). Consider the maximisation problem for firm i, $\underset{p_i^A, p_i^B}{Max}\, \pi_i$ subject to $p_i^A \leq p_i^B$ whose Lagrangean is $L_i = \pi_i + \lambda\left(p_i^B - p_i^A\right)$. Both objective and constraint functions are concave in prices and so the Kuhn Tucker conditions are necessary and sufficient for a solution to this problem. Assuming that the constraint is binding and firm i sets $p_i^A = p_i^B = p_i$ say. $\frac{\partial L_i}{\partial p_i^A} = 0$ and $\frac{\partial L_i}{\partial p_i^B} = 0$ give us $\lambda_i = \frac{\mu\left(p^A - p^B\right) + p^B - 2p_i}{t^A} + \frac{\mu p_i}{t^B} - \frac{1}{n} = \frac{(2+\mu)p_i - p^B}{t^B} - \frac{1}{n}$ such that $p_i = \frac{t^B\left[\left(5t^A - t^B\right)\mu + 4t^A t^B - 2t^B\mu^2\right]}{2n(1+\mu)\left(t^A + t^B\right)} \Rightarrow \lambda = \frac{\left(t^A - t^B\right)\left(5\mu^2 - 8\mu + 4\right) + 2t^B\mu^3}{2n(1+\mu)\left(t^A + t^B\right)}$. Now $\lambda > 0$ such that the constraint binds whenever $\frac{t^A}{t^B} \geq \frac{2\mu^3 + 5\mu^2 + 8\mu + 4}{5\mu^2 + 8\mu + 4} \equiv \bar{t}$ this is true throughout region H since $\bar{t} < 1 + 2\mu$. We know from (iii) that firm i would not find deviate to set $p_i^A = p_i^B$.

Having worked through the four possible deviation options for firm i we have shown that the proposed equilibrium will indeed be optimal in the short run. Further, prices here display the expected properties of being decreasing in the number of firms and increasing in transport costs.

Local pricing only occurs when the difference in transport costs is sufficiently large to counter the affect of the disaffection parameter, or the disaffection parameter is sufficiently small. The statement of the short run equilibrium is completed with the profits.

$$\pi = \begin{cases} \dfrac{4t^A t^B}{n^2\left(t^A + t^B\right)} & \dfrac{t^A}{t^B} \leq 2\mu + 1 \\ \dfrac{t^A + (1+2\mu)t^B}{n^2(1+\mu)} & \dfrac{t^A}{t^B} > 2\mu + 1 \end{cases} \tag{7}$$

Figures 2 and 3 plot the effect of μ on the prices and profits gained in the short run equilibrium. The role of transport costs in measuring the intensity of competition in a market has already been noted. Hence lower transport costs mean lower prices and thus lower profits for firms. This is exactly the same effect as having more stores in the market, higher n. When $\mu = 0$ the standard Salop values for prices and profits result, which are then the intercepts on the figures. Disaffection causes firms to want to reduce the differential between their prices in the two markets. Hence as μ increases the differential falls until eventually it equalises the prices and from then on uniform pricing is adopted. Profits fall until they reach the uniform pricing level, this follows from the narrowing of the price gap towards the uniform price.

Figure 2 here

Figure 3 here

4. LONG RUN

We assume that any entrant opens branches in both markets. It will do this if it is able to make a positive profit from both markets. Rather than assume specific costs of entry in A and B a single fixed cost f of opening two branches in the two markets is assumed. Calculating the long run number of firms is simply a case of solving $\Pi = \pi - f = 0$

Proposition 2: There exists a unique long run equilibrium with:

Uniform Pricing: $n = 2\sqrt{\dfrac{t^A t^B}{f\left(t^A + t^B\right)}}$ **if** $\dfrac{t^A}{t^B} \in \left[1, 1 + 2\mu\right]$

Local Pricing: $n = \sqrt{\dfrac{t^A + (1 + 2\mu)t^B}{f(1 + \mu)}}$ **if** $\dfrac{t^A}{t^B} > 1 + 2\mu$

The proof of this is by rearrangement of $\Pi = 0$ using π from (7). These firm numbers also display the expected properties of being increasing in transport costs and decreasing in the fixed cost f and disaffection parameter μ. This follows because of the short run profit reducing effect of disaffection.

5. SOCIAL OPTIMUM AND COMPARISON

We imagine now that there is a social planner who can choose prices and firm numbers in order to maximise the aggregate surplus. To do this the good will be made available at marginal cost $p^A = p^B = c = 0$ and the planner must choose n to minimise total costs for firms and consumers, where total costs, T, are the sum of all transport and entry costs, $T = fn + \dfrac{t^A + t^B}{4n^2}$. [19]

[19] The consumer travelling furthest from the shop they use in a symmetric equilibrium lives at the midpoint of two outlets, a distance $\dfrac{1}{2n}$ from both. Total transport cost for all consumers living between store i and its furthest away customer in market X will be $\int_0^{\frac{1}{2n}} t^X x dx = \dfrac{t^X}{4n}$. There are two

Proposition 3: There exists a unique socially optimal equilibrium n^S

with $p^A = p^B = c = 0$ and $n^S = \frac{1}{2}\sqrt{\frac{t^A + t^B}{f}}$.

Proof: $\frac{\partial^2 T}{\partial n^2} = \frac{8\left(t^A + t^B\right)}{n^3} > 0$ and so T is strictly convex in n and there is a unique cost minimising number of firms where $\frac{\partial T}{\partial n} = f - \frac{t^A + t^B}{4n^2} = 0 \Rightarrow n^S = \frac{1}{2}\sqrt{\frac{t^A + t^B}{f}}$. ■

The social optimum is also increasing in transport costs since the planner would like more firms to minimise the costs consumers face reaching their nearest store. However because firm costs are also considered, greater fixed costs of entry will reduce the socially optimal number.

Corollary 1: The market equilibrium has more chain stores than is socially optimal.

Proof: First $n^S < n^U \Rightarrow 16t^A t^B > \left(t^A + t^B\right)^2 \Rightarrow \gamma^2 - 14\gamma + 1 < 0$ where $t^A = \gamma t^B$ this is strictly convex in γ with roots at $\gamma_\pm = 7 \pm 4\sqrt{3}$. Since $\gamma \geq 1$ by assumption we can ignore γ_- and focus on the range $\gamma \in [1, \gamma_+]$ in which excessive entry will occur. Uniform pricing is chosen if $\gamma < 2\mu + 1 \equiv \bar{\gamma}$. Noting that $\mu < 2 + 2\sqrt{2}$ then the maximum value of $\bar{\gamma}$ is $5 + 4\sqrt{2} < 7 + 4\sqrt{3}$ this is true throughout and so $\gamma \in [\gamma_-, \gamma_+]$ by assumption. Uniform pricing generates over-branching throughout. Secondly $n^S < n^L$ if $(\mu - 3)\gamma < 3 + 7\mu$. This holds for all $\mu < 3$ and as the maximum

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such segments for each store and n stores in each market. Total costs will then be total costs in market A plus total costs in market B plus the total entry costs fn.

value of μ in region H is approximately 1.58, it follows that this condition holds throughout and there are too many chain stores entering the market. ■

This result confirms that of Salop (1979) continues to hold in the assumed parameter set. Disaffection has a profit reducing effect, whether forcing stores to adopt uniform pricing, or simply "averaging" the prices across markets. This effect naturally drives down the number of firms which enter the market. However the effect is never sufficiently large to bring the number of entrants below that which is socially optimal. That the gap narrows however could be seen as a socially beneficial consequence of the presence of disaffection.

6. CONCLUSIONS

UK food retail, like many other industries, has a small number of imperfectly competitive multimarket firms who sell in markets that are very different in terms of competition faced and demand elasticity. However, rather than vary prices to extract maximum possible surplus, stores actually set national prices. In this paper the observation that to discriminate would ruin the store's image of everyday low pricing (consistent with other studies that show that people care when they are charged more for a good that is available cheaper elsewhere) is used to create a theoretical model of disaffection that has uniform pricing as a possible Nash equilibrium. Here the need for stores to commit to, and be bound by, uniform pricing is removed. Irrespective of the degree of competition in the markets a non empty parameter range is identified in which a chain store would choose to adopt a uniform price.

Focus on non price methods for discrimination, such as facilities offered to families, could offer interesting new takes on store practice. It would also be a useful exercise to endogenise quality into the price discrimination discussion as is considered in Ikeda and Toshimitsu (2010). Further exploration of cases where people prefer firms to overcharge them to the benefit of others, and other means of capturing disaffection remain simpler extensions. However the existence of a parameter set in which national pricing is optimal remains a useful foundation and provides a potential explanation for observed practice in supermarket retail and beyond.

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APPENDIX

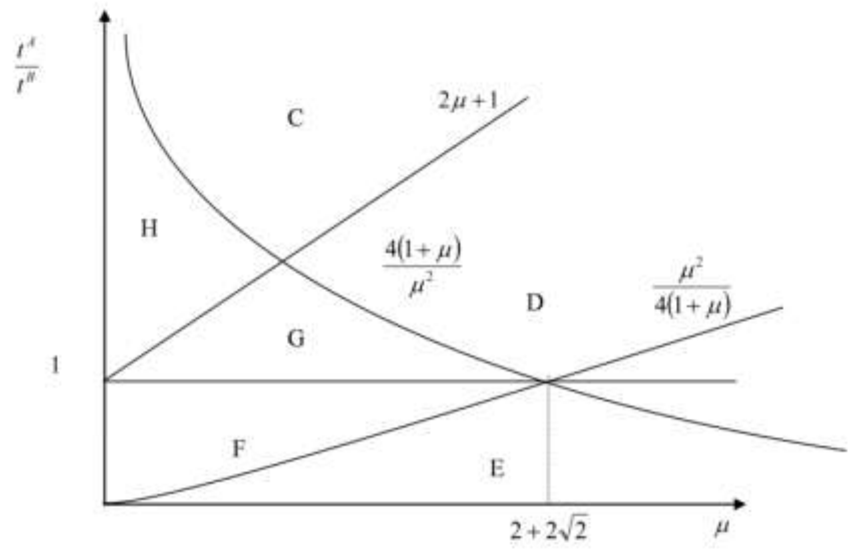

Figure 1 – Parameter Regions

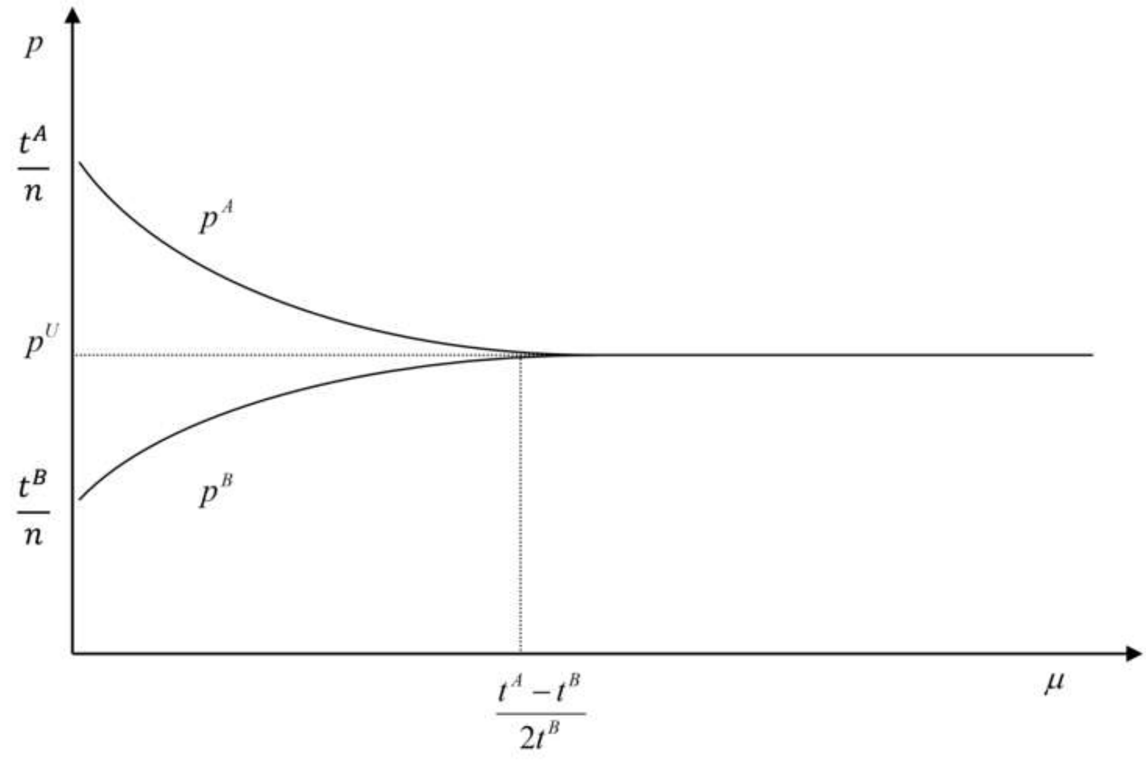

Figure 2 – Effect of Disaffection on Prices

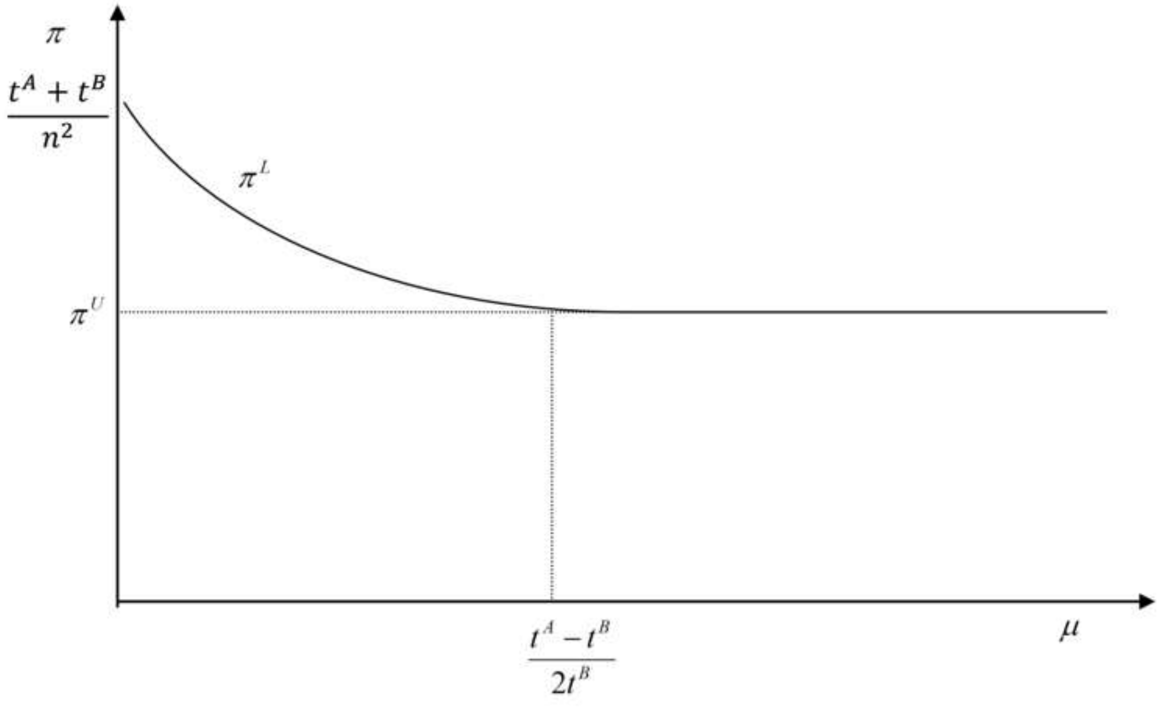

Figure 3 – Effect of Disaffection on Profits

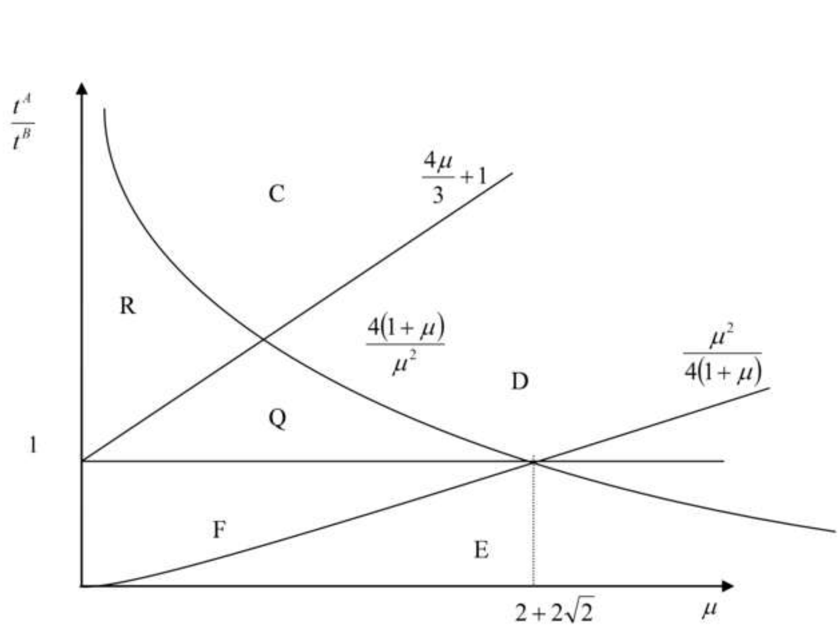

Figure 4 – Parameter Regions in Local Competition

Pareto-Undominated and Socially-Maximal Equilibria in Non-Atomic Games*

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July 1, 2014

Abstract: This paper makes the observation that a finite Bayesian game with diffused and disparate private information can be conceived as a large game with a non-atomic continuum of players. By using this observation as its methodological point of departure, it shows that (i) a Bayes-Nash equilibrium (BNE) exists in a finite Bayesian game with private information *if any only if* a Nash equilibrium exists in the induced large game, (ii) both Pareto-undominated and socially-maximal BNE exist in finite Bayesian games with private information. In particular, it shows these results to be a direct consequence of results for a version of a large game re-modeled for situations where different players may have different action sets.

Keywords: Non-atomic games, saturated probability space, Nash equilibrium, Bayes-Nash equilibrium (BNE), Pareto-undominated equilibrium, socially-maximal equilibrium.

JEL Classification Numbers: C72, C62, D50

*The authors are grateful to Ali Khan, Richard McLean, Marcin Pęski and Yeneng Sun for stimulating conversation and correspondence. Some of the results reported here were presented at the *5th Asian General Equilibrium Theory Workshop*, Xiamen, China, August 4-6, 2008; at the *13th Conference of the Society for the Advancement of Economic Theory* at MINES ParisTech, July 22-26, 2013; at the *AMES*, Taipei, June 19-22, 2014. We thank all participants at these conferences and especially Peter Hammond, Wei He, Kali Rath, Xiang Sun, Nicholas Yannelis, Shmuel Zamir and Yongchao Zhang for useful comments. Yu would like to thank the Office of the Dean of Arts, Ryerson University, for a grant that provided support for the preparation of this article.

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1 Introduction

It is well known that pure strategy Nash equilibria[1] do not necessarily exist in general, and that one could resort to a non-atomic measure space and a restriction of player interdependence to ensure such existence theorems. In particular, one needs to formalize a situation where each player is game-theoretically negligible, and in addition to her own strategy, a player's payoff depends everyone else's strategies. Unlike finite games, the *other* in large (non-atomic) games is no longer a player or a fully delineated group of players, but rather the society or the collectivity that is the formalized subject of the game. The existence theory of Nash equilibria in such large games is now well-understood.[2] Once Nash equilibria are shown to exist in such a game, many refinements of such equilibria can be discerned and usefully categorized; see, for example, Rath [32, 33] for some refinements such as perfect and proper equilibria.

Among Nash equilibria, it is possible that all players can jointly deviate from a particular equilibrium outcome to choose another equilibrium at which they are all better off. This suggests a search for a refinement of Nash equilibrium that is not Pareto dominated by any other Nash equilibria. We call such refined Nash equilibria *Pareto-undominated* Nash equilibria.[3] To be more specific, a Pareto-undominated Nash equilibrium admits no other Nash equilibrium that (a) makes no player worse off, and (b) makes at least one player strictly better off. This refinement has been widely used by applied economists.[4] However, in general, even if there exists a Nash equilibrium, a Pareto-undominated Nash equilibrium may not exist in a game. For example, in a two-player game, with each player's action set as the open unit interval [0,1), and with payoffs equating their choices only when the two choices are identical, and zero otherwise, there is a continuum of Nash equilibria, none of which are Pareto-undominated. This paper addresses the question as to when a Pareto-undominated Nash equilibrium exists.

There is by now a clear understanding that in the theory of large games, even if externalities or statistical summaries are formalized as an integral of societal responses, the action sets have to have enough of a structure that individual responses can be aggregated, which is to say, can be integrated. This requires a non-trivial extension of integration theory even in the case where the action sets are countably-infinite, leave alone sets of uncountable cardinality; see Khan et al. [17] for example. There are several papers in the literature of large games that address the issue of existence of Pareto-undominated equilibria, but they do so in a setting where the externalities are formulated in a finite dimensional space.[5] This is a rather severe limitation. And so the question is

[1]Unless specified otherwise, all references to Nash equilibria in this paper refer to pure strategy Nash equilibria even where the term "pure strategy" is not used.

[2]See the survey and the references in Khan and Sun [21].

[3]This is different from the notion of an undominated Nash equilibrium (e.g., Kultti and Salonen [24] and Salonen [36]). The latter refers to a Nash equilibrium where none of the players use a weakly dominated strategy; also see Remark 1 below for a discussion of this notion in large games.

[4]See Yi [39] for a discussion on this.

[5]See Le Breton and Weber [25], Codognato and Ghosal [7] and Balder [4]. To be more specific, the set of actions

whether there exist Pareto-undominated Nash equilibria in large games where statistical summaries are formulated as distributions of actions. This paper is therefore to fill the gap in the non-trivial case where integrals and distributions diverge, which is to say when action sets are not finite.

Moreover, due to the need of applications,[6] the theory of large games has well gone beyond a framework in which a player's choice among a finite number of actions depends on a statistical summary, be it an *average* or a *distribution*, of the choices of everyone else. Recently, Khan et al. [20] consider situations based on a bio-social typology so that the notion of player interdependence is broadened to include a dependence on both characteristics and traits which is emphasized in the social identity literature as in Akerlof and Kranton [1]. Such a reformulation[7] covers conventional large games where statistical summaries are formulated as distributions of actions. In this paper, we generalize the setting of Khan et al. [20] by allowing players to have heterogeneous (compact) action sets to fit the need of many economic analysis. Then we show that not only Nash equilibria exist, but also Pareto-undominated Nash equilibria exist in such a setting. Furthermore, we show that if payoffs in the game are uniformly integrable, there also exist *socially-maximal* equilibria (a refinement of Pareto-undominated Nash equilibria) under which the aggregated payoff of all players is no less than any aggregated payoffs under other Nash equilibria.

Shifting to a different register, one of finite Bayesian games of incomplete information, rather than that of large games of complete information, it is by now well-understood that Bayes-Nash equilibria[8] (henceforth, BNE) exist when incomplete information is modeled as being *diffused* and *disparate*.[9] And so, given that non-atomic measure spaces play a prominent role also in the theory of Bayesian games with incomplete information, there has been a long-held view that the two theories are intimately related and one therefore ought to be able to go from one to the other.[10] This intuition has not really been pinned down in the form of a precise theorem,[11] and a traceable analytical engine that can be used for future investigation. Most of the papers simply remained

in large games considered by Le Breton and Weber (1997) is finite, and the externalities induced by strategy profiles in large games considered by Codognato and Ghosal (2002) and Balder (2003) are restricted to an n-dimensional Euclidean space. As such, these results are dependent on finite-dimensional integration, which leads to a theory that does not carry over to an *infinite* dimensional setting.

[6]In addition to the search models considered in Rauh [34, 35], see Guesnerie and Jara-Moroni [12] for a discussion on applications of large games to the frameworks of partial equilibrium, general equilibrium, finance, and macro-economics.

[7]For large but finite games, see Kalai [15] for the case with finitely many types/traits where the interdependence assumption is called semi-anonymity.

[8]As indicated in Footnote 1, "pure strategy" is generally dropped in this paper for convenience.

[9]In addition to Radner and Rosenthal [30] for the formalization of these intuitions in the framework of Harsanyi [13], see Aumann et al. [3].

[10]Mas-Colell [27, Remark 3] suggests that the existence of BNE in finite games with diffused information can be deduced as a consequence of the existence result of equilibria in its induced large game. More recently, Balder [5, Section 4] also demonstrates that the existence result of Nash equilibria (which involves finite-dimensional integration) in a so-called internal-external *form* of a large non-atomic game can be used to establish the existence of BNE in Milgrom-Weber type game when actions are *finite*; see Footnote 5 above and Footnotes 12 and 15 below.

[11]Fu [10, Chapter 5] is an important exception. There the connection between equilibria in large games with partitions of players and finite-player Bayesian games with private and public information (a generalization of both Radner-Rosenthal and Milgrom-Weber type of games) is established.

satisfied with the fact that the two literatures, those of large games and of Bayesian games with diffused and disparate information share similar analytical techniques in the proofs of the existence of pure-strategy equilibria. In both classes of non-atomic games: for models with finite actions, the existence of equilibria can be obtained through purification by the Dvoretzky-Wald-Wolfowitz purification principle;[12] for models with countable actions, the existence of equilibria in a non-atomic game can be proved through BV marriage lemma;[13] and for a recent development of non-atomic games with arbitrary compact metric action spaces,[14] saturated spaces are used to characterize the existence of pure strategy equilibrium. Because of these similarities, one could then ask whether the existence results of Pareto-undominated and socially-maximal equilibria hold in Bayesian games with incomplete information.

Towards this end, we show that, in fact, one can apply all the results that we establish for a non-atomic large game directly to a Bayesian game with diffused and disparate private information. This connection between the two literatures is surely of interest in itself from a methodological point of view, but also allows us to resolve conclusively the issue of the existence of Pareto-undominated and socially-maximal BNE in Bayesian games—a resolution obtained as a *byproduct* that sits squarely on the results for large non-atomic games. The trick[15] simply lies in that we can treat a *real* player together with her type in a Bayesian game with diffused and disparate private information as an *artificial* player, and use the real player's name as the trait of the artificial player in the induced large game. More specifically, with the standard diffuseness and mutual independence assumptions, we transfer a Bayesian game with private information to a large game and establish that a BNE exists in the original Bayesian game *if and only if* a Nash equilibrium exists in the induced large game.[16] We then propose notions of Pareto-undominated and socially-maximal BNE and show that existence of such BNE can also be obtained through the corresponding results in the induced large game.

The paper is rather simply organized in terms of two substantive sections: Section 2 focuses on a reformulated large non-atomic game, and Section 3 on a Bayesian game with diffused and

[12]See, for example, Schmeidler [37] on large games and Radner and Rosenthal [30] and Milgrom and Weber [28] on Bayesian games. In Radner and Rosenthal [30, Footnote 3], the authors write: "The method of proof of Theorem 1 was suggested by Schmeidler (1973). It is also reminiscent of Dvoretzky et al. (1950)." For more details on how to use DWW purification principle to non-atomic games with finite actions, see Khan et al. [19].

[13]See Khan et al. [17] and Khan and Sun [21] for example.

[14]See, for example, Keisler and Sun [16] and Khan et al. [20] on large games and Khan and Zhang [22] and He and Sun [14] on Bayesian games.

[15]The prototype of this trick, in the context of a Bayesian game with finite types, players and actions, is called "a third model of Bayesian games" suggested by Selten in Harsanyi [13, Page 177] where an artificial induced game with a larger number of players is used to deal with BNE in the original Bayesian game. Such a transformation is not that clear in non-atomic games due to the structure of externalities that are involved. In fact, it is important for the reader to appreciate that the conventional large game model (where the externalities are just distributions on actions) is *not suitable* to carry out this transformation. For this technical point, we refer the reader to Theorem 3 and its proof.

[16]It is worth pointing out that it does not say that we can also explicitly transfer any large game into a finite Bayesian game, and establish that a Nash equilibrium exists in the large game *if and only if* a BNE exists in the induced Bayesian game. This is still an open question.

disparate private information. In both sections, under some standard assumptions, we show respectively the existence results of Nash and Bayes-Nash equilibria, and more importantly, their Pareto-undominated and socially-maximal counterparts. Section 4 concludes the paper. All proofs are provided in Appendix.

2 Large Games

In a conventional large (non-atomic) game, an abstract non-atomic probability space is used to denote the space of players, and a compact metric space is used to represent a common action space where the common action space is then used to build the space of externalities (distributions on action space) and the space of payoffs (continuous functions on the product space). Due to the need for a rich space of player characteristics which consists of both traits and payoffs, Khan et al. [20] generalize the conventional large game into a formulation[17] that incorporates traits and allows externalities to be joint distributions of actions and traits For convenience, we call such a game as a large games with traits (henceforth, LGT). We now consider a generalized LGT model by allowing that different players may have heterogeneous strategy sets.

2.1 The Model

Let a non-atomic probability space $(I, \mathcal{I}, \lambda)$ be the space of players, a complete and separable metrizable (Polish) space T the space of traits, and a separable complete metric space space A the space of actions for all players. To allow heterogeneous strategy sets for different players, we allow that each player $i \in I$ chooses her own actions from $D(i) \in A$ where D is a nonempty, compact-valued and measurable correspondence.[18] A *(pure) strategy profile* is a measurable selection[19] of D, which specifies a pure strategy for each player.

A player's *characteristics* consists of two components: *trait* and *payoff*. The *trait* function is a measurable function $\alpha : I \to T$ which assigns each player $i \in I$ an exogenously given trait $\alpha(i)$. Let $\mathcal{M}(T \times A)$ be the space of Borel probability distributions on $T \times A$ endowed with the topology of weak convergence of probability measures.[20] The statistical summary under α and a given strategy profile f is therefore $\lambda(\alpha, f)^{-1}$. Let

$$\mathscr{D}_D^\alpha = \{\lambda(\alpha, f)^{-1} : f \text{ is a measurable selection of } D\}$$

[17]Also see Qiao and Yu [29] for the continuity consideration of such games.

[18]Recall that a correspondence D from $(I, \mathcal{I}, \lambda)$ to A is said to be *measurable* if for each closed subset C of A, the set $D^{-1}(C) = \{i \in I : D(i) \cap C \neq \emptyset\}$ is measurable in $\mathcal{I}$.

[19]A function f from I to A is said to be a *measurable selection* of D if f is measurable and $f(i) \in D(i)$ for all $i \in I$. The classical Kuratowski-Ryll-Nardzewski Theorem (e.g., Aliprantis and Border [2, Theorem 18.13]) guarantees that there exists a measurable selection of D.

[20]Unless otherwise specified, any topological space discussed in this paper is tacitly understood to be equipped with its Borel σ-algebra (i.e., the σ-algebra generated by the family of open sets) and the measurability is defined in terms of it.

be the set of all statistical summaries. It is clear that $\mathscr{D}_D^\alpha \in \mathcal{M}(T \times A)$. The *payoff* function of player $i \in I$ is given by $v(i, \cdot, \cdot) : D(i) \times \mathscr{D}_D^\alpha \to \mathbb{R}$. Assume that $v(i, \cdot, \cdot)$ is continuous on $D(i) \times \mathscr{D}_D^\alpha$ for all $i \in I$, and $v(\cdot, \cdot, \tau)$ is a measurable function on the graph of D for all $\tau \in \mathscr{D}_D^\alpha$.

An LGT that allows heterogeneous strategy sets for players is thus summarized by

$$\mathcal{G} = ((I, \mathcal{I}, \lambda), T, (A, D), (\alpha, v))\,,^{21}$$

where all objects are described as above.

2.2 Existence of Nash Equilibria

A *Nash equilibrium* of an LGT $\mathcal{G}$ is simply a strategy profile that satisfies Nash property. Formally, it is defined as below:

Definition 1. *A Nash equilibrium of an LGT $\mathcal{G}$ is a strategy profile f^* of the game such that for λ-almost all $i \in I$,*

$$v\left(i, f^*(i), \lambda(\alpha, f^*)^{-1}\right) \geq v\left(i, a, \lambda(\alpha, f^*)^{-1}\right) \text{ for all } a \in D(i).$$

To obtain an existence result that holds for general actions, we need to rely on the following notion of a *saturated* probability space which is recently introduced into the theory of large games:[22]

Definition 2. *A probability space is said to be almost-countably generated if its σ-algebra can be generated by a countable number of subsets together with the null sets; otherwise, it is not almost-countably generated. A probability space $(I, \mathcal{I}, \lambda)$ is saturated if it is nowhere almost-countably generated, in the sense that for any subset $S \in \mathcal{I}$ with $\lambda(S) > 0$, the restricted probability space $(S, \mathcal{I}^S, \lambda^S)$ is not almost-countably generated, where $\mathcal{I}^S := \{S \cap S' : S' \in \mathcal{I}\}$ and λ^S is the probability measure re-scaled from the restriction of λ to $\mathcal{I}^S$.*

It follows that a saturated probability space is non-atomic from the definition. We are now ready to present our result on the existence of Nash equilibria in an LGT.

Theorem 1. *There exists a Nash equilibrium in an LGT $\mathcal{G}$, provided that (i) T and A are both countable spaces, or (ii) $(I, \mathcal{I}, \lambda)$ is a saturated probability space.*

[21] If the action correspondence is constant-valued, then this game reduces to the model with a common action set in Khan et al. [20].

[22] It has been shown, for example, in Keisler and Sun [16] and Khan et al. [20] that the saturated assumption on the space of players is not only *sufficient* but also *necessary* towards the existence of a Nash equilibrium in a large game, with or without traits when action set is uncountable or the induced distribution of the space of traits is non-atomic. We refer the reader to Khan et al. [20] for a detailed discussion on this.

2.3 Pareto-Undominated and Socially-Maximal Nash Equilibria

Given that the existence of Nash equilibria in $\mathcal{G}$ is not vacuous, we can refine Nash equilibria to a set whose elements are immune to grand coalitional deviations to other equilibria in general. We first define Pareto dominance.

Definition 3. *In an LGT $\mathcal{G}$, a strategy profile f is Pareto dominated by a strategy profile f' if for λ-almost all $i \in I$,*

$$v(i, (f'(i), \lambda(\alpha, f')^{-1}) \geq v(i, (f(i), \lambda(\alpha, f)^{-1}),$$

with the strict inequality for a set of players with positive measure. A Nash equilibrium of $\mathcal{G}$ is a Pareto-undominated Nash equilibrium if it is not Pareto dominated by any other Nash equilibria of the game.

The next result is to address the existence of a Pareto-undominated Nash equilibrium in an LGT.

Theorem 2. *Under the hypotheses of Theorem 1, a Pareto-undominated Nash equilibrium exists.*

We next define another Pareto-undominated refinements of Nash equilibria—socially-maximal Nash equilibrium in an LGT.

Definition 4. *In an LGT $\mathcal{G}$, a strategy profile f is dominated in social welfare by a strategy profile f' if for λ-almost all $i \in I$,*

$$\int_I v(i, (f'(i), \lambda(\alpha, f')^{-1}) d\lambda > \int_I v(i, (f(i), \lambda(\alpha, f)^{-1}) d\lambda.$$

A Nash equilibrium of $\mathcal{G}$ is said to be a socially-maximal Nash equilibrium if it is not dominated in social welfare by any other Nash equilibria of $\mathcal{G}$.

The domination in social welfare does not necessarily imply Pareto domination, but Pareto domination implies domination in social welfare. Hence, the contrapositive of the latter statement gives that any undominated strategy profile in social welfare is a Pareto-undominated strategy profile. Thus, it is of interest to ask if we can strengthen Theorem 2 to assert the existence of a socially-maximal Nash equilibrium in the same setting. Our last result in this section is on this and the following uniform integrability condition is needed:

Assumption 1. *There is a real-valued integrable function ϕ on $(I, \mathcal{I}, \lambda)$ such that for λ-almost all $i \in I$, $|v(i, a, \tau)| \leq \phi(i)$ for all $a \in D(i)$ and all $\tau \in \mathscr{D}_D^{\alpha}$.*

Proposition 1. *If Assumption 1 holds, a socially-maximal Nash equilibrium exists under the hypotheses of Theorem 1.*

We conclude this section by the following remark.

Remark 1. As pointed out in Footnote 3, the notion of undominated Nash equilibrium is different from the notion of Pareto undominated Nash equilibrium. As shown in Salonen [36], there does not exist in general an undominated Nash equilibrium in a game, and it is therefore of interest to ask whether an undominated Nash equilibrium in the LGT considered here. The answer is affirmative. One can simply modify the proof of Theorem 2 along the lines of the argument in Rath [33, Section 5] to obtain the existence of an undominated Nash equilibrium directly. We skip the details here and refer the reader to his paper for an extended discussion. Furthermore, examples of LGTs can be constructed to show that an undominated Nash equilibrium is not necessarily Pareto undominated, or vice versa.

3 Bayesian Games with Private Information

We now consider Bayesian games with private information in this section. We first present a basic Bayesian framework and some regularity assumptions in Section 3.1. Under those assumptions, we show in Section 3.2 that a Bayesian game can be transferred into an artificial LGT such that the Bayesian game has a BNE *if and only if* the induced LGT has a Nash equilibrium. Based on this result, Section 3.3 is devoted to address the existence results of Pareto-undominated and socially-maximal BNE in the Bayesian game.

3.1 The Model

Consider a Bayesian game with private information Γ as follows:

(i) A finite index set T with $|T|$ as its cardinality represents the *set of players*.[23]

(ii) For each player $t \in T$, measurable spaces $(Z_t, \mathcal{Z}_t)$ and $(X_t, \mathcal{X}_t)$ represent the *space of strategy-relevant private information* and the *space of payoff-relevant private information* of player t respectively. Let $(\Omega, \mathcal{F})$ be the product measurable space $(\prod_{t\in T}(Z_t \times X_t), \prod_{t\in T}(\mathcal{Z}_t \times \mathcal{X}_t))$, and μ a probability measure on $(\Omega, \mathcal{F})$. For a point $\omega = (z_1, x_1, \cdots, z_l, x_l, \cdots) \in \Omega$, define the coordinate projections $\zeta_t(\omega) = z_t, \chi_t(\omega) = x_t$. The random mappings $\zeta_t(\omega)$ and $\chi_t(\omega)$ are interpreted respectively as *the private information* of player t related to her action and payoff when nature plays ω.

(iii) For each player $t \in T$, a separable complete metric space space A_t represents her *action space* and a measurable correspondence L_t from Z_t to A_t represents the *action correspondence* of player t. This is to say, player t first observes some realization, say $z_t \in Z_t$, of the random element $\zeta_t(\omega)$, then chooses her own action from a nonempty compact subset $L_t(z_t)$ of A_t. Let $\mathcal{A} = \prod_{m\in T} A_m$. Without loss of generality, we can assume that each A_t is a subspace of a complete separable metric

[23]There is an abuse of notation here as we use T as the space of traits in Section 2. But this decision to let T denote the set of players simply because later on we will use T as the space of traits in the induced LGT of Γ in Section 3.2. In addition, instead of using a number, we use $|T|$ to denote the cardinality of T in order to accommodate the possible generalization of our result to allow countably many players. See Remark 1 below.

space A and $(A_m)_{m\in T}$ are disjoint.[24].

(iv) For every player $t \in T$, her *payoff* function is a function $u_t : \mathcal{A} \times X_t \to \mathbb{R}$ that satisfies: (i) for any fixed $x_t \in X_t$ $u_t(\cdot, x_t)$ is a continuous function on $\mathcal{A}$, and (ii) for any fixed $\mathbf{a} \in \mathcal{A}$,[25] $u_t(\mathbf{a}, \cdot)$ is a measurable function on $(X_t, \mathcal{X}_t)$.

In summary, a Bayesian game with private information is given by

$$\Gamma = \big(T, ((Z_t, \mathcal{Z}_t), (X_t, \mathcal{X}_t), (A_t, L_t), u_t)_{t\in T}, \mu\big),$$

with each object described as in (i)-(iv) above.

For any player t, let $\text{meas}(Z_t, L_t)$ be the set of measurable mappings f from $(Z_t, \mathcal{Z}_t)$ to A_t such that $f(z_t) \in L_t(z_t)$ for each $z_t \in Z_t$. A *pure strategy* for player t is thus an element g_t of $\text{meas}(Z_t, L_t)$. A *strategy profile* g is a collection $(g_m)_{m\in T}$ that specifies a pure strategy for each player. In order to define expected payoffs below and regular conditional expectations later, we make the following assumption:

Assumption 2. *For each player t, there is a real-valued integrable function φ_t on $(\Omega, \mathcal{F}, \mu)$ such that for μ-almost all $\omega \in \Omega$, $|u_t(\mathbf{a}, \chi_t(\omega))| \le \varphi_t(\omega)$ holds for all $\mathbf{a} \in \mathcal{A}$.*

It is now clear that once a strategy profile g is played, the expected payoff of player t can be expressed as:

$$U_t(g) = U_t((g_m)_{m\in T}) = \int_{\omega\in\Omega} u_t\big((g_t(\zeta_t(\omega)))_{t\in T}, \chi_t(\omega)\big)\, \mu(d\omega).$$

We are ready to define a BNE of Γ.

Definition 5. *In a Bayesian game with private information Γ, a BNE is a strategy profile $g^* = (g^*_m)_{m\in T}$ such that for all $t \in T$,*

$$U_t(g^*) \ge U_t(g_t, g^*_{-t}), \text{ for all } g_t \in \text{meas}(Z_t, L_t).$$

In order to study the existence of BNE in a Bayesian game with private information, the following standard regularity conditions are often used:

Assumption 3. *For each player $t \in T$ in Γ, (i) the distribution $\mu\zeta_t^{-1}$ of ζ_t is non-atomic, and, (ii) random elements $\{\zeta_m : m \ne t\}$ together with the random element $\xi_t \equiv (\zeta_t, \chi_t)$ form a mutually independent set.*

[24] If $(A_t, d_t)_{t\in T}$, where $(d_t)_{t\in T}$ are the metrics, are indeed from different metric spaces, it is easy to construct another complete separable metric space (A, d) encompassing them as follows: if $(A_t, d_t)_{t\in T}$ are disjoint, we can just define (A, d) by letting $A = \cup_{t\in T} A_i$ and $d|_{A_t} = d_t$ for $t \in T$ and $d(a_t, a_m) = +\infty$ if $a_t \in A_t$, $a_m \in A_m$, and $t \ne m$. It is straightforward to check that (A, d) is also a complete separable metric space and $(A_t, d_t)_{t\in T}$ are its subspaces. If $(A_t, d_t)_{t\in T}$ are not disjoint, we can add an index to each A_t to distinguish them.

[25] Throughout the paper, we use the boldface $\mathbf{a}$ to denote an action profile from the set of action profiles $\mathcal{A}$ and the usual a a single action from set A. Moreover, we use the following (conventional) notation: $A_{-t} = \prod_{m\in T, m\ne t} A_m$, $Z_{-t} = \prod_{m\in T, m\ne t} Z_m$, $g = (g_t, g_{-t})$ and $\mathbf{a} = (a_t, a_{-t})$ for $\mathbf{a} \in \mathcal{A}$ throughout the paper. Without any ambiguity, we also abbreviate $\prod_{j\in I, j\ne i}$ to $\prod_{j\ne i}$.

In the assumption above, part (i) means that each player's strategy-relevant private information is diffused, and (ii) means that each player's private information is independent of all other players' strategy-relevant private information in the game. Thus, we call Γ a Bayesian game with *diffused and disparate* private information if Assumption 3 is satisfied in Γ.

3.2 Existence of BNE: Transforming a Bayesian Game to an LGT

As discussed in the introduction, large games and Bayesian games with diffused and disparate information share some similar analytical techniques towards the existence of equilibria. We now elaborate this observation and show the *exact equilivalence* connection of equilibrium concepts in these two frameworks.

Let Γ be a Bayesian game with private information that satisfies Assumptions 2 and 3. We next show that Γ can be transferred into an artificial LGT $\mathcal{G}^\Gamma$, and establish that Γ has a BNE *if and only if* the induced LGT $\mathcal{G}^\Gamma$ has a Nash equilibrium.

We first construct the space of players of the LGT $\mathcal{G}^\Gamma$. The basic idea is to treat a combination of a player t and one of its types in Γ as an "artificial" player in the LGT $\mathcal{G}^\Gamma$ such that this artificial player has trait t. Towards this end, let $I_t \equiv Z_t \times \{t\}$ and $\mathcal{I}_t \equiv \mathcal{Z}_t \otimes \{t\}$. Define a probability measure λ_t on $(I_t, \mathcal{I}_t)$ such that $\lambda_t(B_t \times \{t\}) = \mu\zeta_t^{-1}(B_t)$ for any $B_t \in \mathcal{Z}_t$. Clearly, $(I_t, \mathcal{I}_t, \lambda_t)$ is a well-defined probability space. Since t is arbitrarily chosen from T, we now allow t to be any element of T and repeat all steps above. Let $I^\Gamma = \bigcup_{t\in T} I_t$ and $\mathcal{I}^\Gamma = \{\bigcup_{t\in T} C_t : C_t \in \mathcal{I}_t\}$. For any $C \in \mathcal{I}^\Gamma$, let

$$\lambda^\Gamma(C) = \sum_{t\in T} \frac{1}{|T|}\lambda_t(C \cap I_t) \tag{1}$$

We need to show that $(I^\Gamma, \mathcal{I}^\Gamma, \lambda^\Gamma)$ is non-atomic so that it can be a well-defined space of players for an LGT.

Lemma 1. *$(I^\Gamma, \mathcal{I}^\Gamma, \lambda^\Gamma)$ is a non-atomic probability space.*

Let $(I^\Gamma, \mathcal{I}^\Gamma, \lambda^\Gamma)$ be the *space of player* and T be the *space of traits* of $\mathcal{G}^\Gamma$. In addition, let the *trait* function of $\mathcal{G}^\Gamma$, $\alpha^\Gamma : (I^\Gamma, \mathcal{I}^\Gamma, \lambda^\Gamma) \to T$ and the action correspondence $D^\Gamma : I \to A$ be such that $\alpha^\Gamma(i) = t$ and $D^\Gamma(i) = L_t(\text{Proj}_{Z_t}(i))$ if $i \in I_t$ for some $t \in T$, where Proj_{Z_t} denotes the projection mapping from I_t to Z_t. Basically, this is to say, for any artificial player $i = (z, t) \in I^\Gamma$, her given trait is specified by player t's index (or name) t in the original Bayesian game Γ, and all available actions to her are specified by $D^\Gamma((z, t))$, which is the available action set $L_t(z)$ for player t with type z in the original game Γ.

We only need to construct *payoff* functions of $\mathcal{G}^\Gamma$.

Towards this end, we first fix $t \in T$ in Γ. By Assumption 3, the expected payoff for player t in

Γ under a strategy profile g can be expressed as

$$U_t(g_t, g_{-t}) = \int_{A_{-t}} \int_\Omega u_t\left(\left(g_t\left(\zeta_t(\omega)\right), a_{-t}\right), \chi_t(\omega)\right) \mathrm{d}\mu \, \mathrm{d}\Pi_{m \neq t}(\mu\zeta_m^{-1})g_m^{-1}.$$

Furthermore, by Assumption 2, we can apply Dynkin and Evstigneev [8, Theorem 2.1] to assert that there exists a function $V_t : \mathcal{A} \times Z_t \to \mathbb{R}$, such that $V_t(\mathbf{a}, \zeta_t(\omega)) \equiv \mathrm{E}\{u_t(\mathbf{a}, \chi_t(\omega))|\zeta_t(\omega)\}$ is the regular conditional expectation of $u_t(\mathbf{a}, \chi_t(\omega))$ under the sub-σ-algebra of $\mathcal{F}$ generated by ζ_t. This is to say, for any measurable set $W \in \mathcal{Z}_t$, we have

$$\int_{\{\omega \in \Omega : \zeta_t(\omega) \in W\}} u_t(\mathbf{a}, \chi_t(\omega)) d\mu(\omega) = \int_{z_t \in W} V_t(\mathbf{a}, z_t) d\mu\zeta_t^{-1}(z_t).$$

Moreover, by Dynkin and Evstigneev [8, Theorem 2.2], $V(\cdot, z_t)$ is continuous and bounded on $\mathcal{A}$ for $\mu\zeta_t^{-1}$-almost all $z_t \in Z_t$. Without loss of generality, we can assume for all $z_t \in Z_t$, $V(\cdot, z_t)$ is continuous and bounded on $\mathcal{A}$.[26]

Next, for any $m \in T$, let $\mathscr{D}_{L_m} = \{(\mu\zeta_t^{-1})s^{-1} : s \text{ is a measurable selection of } L_m\}$. Let $\bar{V}^t : Z_t \times A_t \times \prod_{m \in T} \mathscr{D}_{L_m} \longrightarrow \mathbb{R}$ be a function such that for any $z_t \in Z_t$, $a_t \in A_t$ and $(\tau_m)_{m \in T} \in \prod_{m \in T} \mathscr{D}_{L_m}$,

$$\bar{V}_t(z_t, a_t, (\tau_m)_{m \in T}) = \int_{A_{-t}} V_t(a_t, a_{-t}, z_t) \mathrm{d} \prod_{m \neq t} \tau_m,$$

For any given a_t, z_t, this $\bar{V}_t(z_t, a_t, (\tau_m)_{m \in T})$ is simply the expected payoff to player t when she takes the action a_t, nature ąřplaysąś ω but only $z_t = \chi_t(\omega)$ is revealed to her, and any other player $m \neq t$ plays the mixed action τ_m. We shall use this expected payoff function $\bar{V}_t(z_t, \cdot, \cdot)$ for player t with a given type $z_t \in Z_t$ to construct the payoff function for the artificial player (z_t, t) in $\mathcal{G}^\Gamma$. Namely, for any artificial player $i \in I_t$, let $\bar{v}^\Gamma(i, \cdot, \cdot) : D^\Gamma(i) \times \prod_{m \in T} \mathscr{D}_{L_m}$ be a function such that for all $a \in D^\Gamma(i)$ and $(\tau_m)_{m \in T} \in \prod_{m \in T} \mathscr{D}_{L_m}$,

$$\bar{v}^\Gamma(i, a, (\tau_m)_{m \in T}) = \bar{V}_t(\mathrm{Proj}_{\,Z_t}(i), a, (\tau_m)_{m \in T}).$$

As t is arbitrarily chosen, we now allow that t varies. Let $\Phi : \mathscr{D}_{D^\Gamma}^{\alpha^\Gamma} \to \prod_{m \in T} \mathscr{D}_{L_m}$ be a mapping such that for any $\tau \in \mathscr{D}_{D^\Gamma}^{\alpha^\Gamma}$,

$$\Phi(\tau) = \{\tau^m\}_{m \in T},$$

where for each $m \in T$, $\tau^m(B) = \frac{\tau(\{m\} \times B)}{\lambda^\Gamma(\alpha^\Gamma)^{-1}(m)}$ for all $B \in \mathcal{B}(A)$. We are now ready to construct payoff functions for *all players* in the induced LGT $\mathcal{G}^\Gamma$ from Γ:

For any player $i \in I^\Gamma$, let her payoff function be $v^\Gamma(i, \cdot, \cdot) : D^\Gamma(i) \times \mathscr{D}_{D^\Gamma}^{\alpha^\Gamma} \to \mathbb{R}$ such that for all $a \in D^\Gamma(i)$ and $\tau \in \mathscr{D}_{D^\Gamma}^{\alpha^\Gamma}$,

$$v^\Gamma(i, a, \tau) = \bar{v}^\Gamma(i, a, \Phi(\tau)).$$

[26]In fact, for almost all $z_t \in Z_t$, $V(\cdot, z_t) \leq \tilde{h}(z_t)$, with $\tilde{\varphi}(\zeta_t) = E[\varphi|\zeta_t]$, where φ is the function in Assumption 2.

All elements in the LGT $\mathcal{G}^\Gamma$ have been specified. We now present our main result on a necessary and sufficient condition for the existence of BNE in Γ.

Theorem 3. *Suppose that a Bayesian game with private information Γ satisfies Assumptions 2 and 3. Then, its induced game*

$$\mathcal{G}^\Gamma = \{(I^\Gamma, \mathcal{I}^\Gamma, \lambda^\Gamma), T, (A, D^\Gamma), (\alpha^\Gamma, v^\Gamma)\}$$

with all elements specified as above, is an LGT that is defined in Section 2.1 and satisfies Assumption 1. Moreover, Γ has a BNE if and only if the induced LGT $\mathcal{G}^\Gamma$ has a Nash equilibrium.

With Theorem 3, the following result on the existence of BNE in Γ can be viewed as a direct corollary of Theorem 1.

Corollary 1. *If Γ is a Bayesian game with private information that satisfies Assumptions 2 and 3, then there exists a BNE in Γ, provided that either of the following two conditions holds: (i) A is countable, or (ii) $(Z_t, \mathcal{Z}_t, \mu\zeta_t^{-1})$ is a saturated probability space for every $t \in T$.*

The following three remarks conclude our discussion on the existence of BNE in a Bayesian game with diffused and disparate information.

Remark 2. It is worth pointing out that even if $|T|$, the number of players, in Γ is countable (i.e., countably infinite, or, finite) instead of being finite, statements in Theorem 3 and Corollary 1 still hold. This observation itself is new in the literature as it allows us to obtain the existence result of BNE in a such a Bayesian game that allows *countable* many players. To see this, we can allow $T = \mathbb{N}$, the set of all natural numbers if T is countably infinite. Then, one only needs to replace (1) by

$$\lambda^\Gamma(C) = \begin{cases} \sum_{t\in T} \frac{1}{|T|}\lambda_t(C \cap I_t), & \text{if } T \text{ is finite;} \\ \sum_{t\in T} \frac{1}{2^t}\lambda_t(C \cap I_t), & \text{if } T = \mathbb{N}. \end{cases}$$

It is easy to check that all proofs in the Appendix still apply.

Remark 3. Condition (i) or (ii) in Corollary 1 is needed to guarantee the existence of a BNE. To see this, let T be finite and for each t and all $z_t \in Z_t$, let $L_t(z_t) = A_t$, a compact metric space. In such a setting, Khan et al. [18] show that if spaces of strategy-relevant private information are Lebesgue measure spaces and action spaces are uncountable, then BNE may not exist at all; Loeb and Sun [26] show that if spaces of strategy-relevant private information are Loeb measure spaces (and hence, saturated) and action spaces are compact metric spaces, then a BNE does exist; results in Khan and Zhang [22] show that (ii) is not only *sufficient* but also *necessary* for the existence of a BNE with the action space being uncountable.

Remark 4. Assumption 3 is standard in the study of the existence of BNE in Bayesian games with diffused and disparate private information in the literature; see Radner and Rosenthal [30], Khan and Sun [21, Section 4.1] and Yu and Zhang [40], for example. The setting of games in Corollary 1 and Remark 2 is more general than any of these papers. Moreover, our method differs from them methodologically: The existence result on BNE in games with diffused and disparate private information is simply a direct consequence and application of Theorem 1. Such a method is also valid to guarantee the existence of BNE in a more general Bayesian game with both *private* and *public* information (as the one initiated by Fu et al. [11] and further studied in Khan and Zhang [22]), which covers both Radner-Rosenthal and Milgrom-Weber formulations; see Footnote 27.

3.3 Pareto-Undominated and Socially-Maximal BNE

In this subsection, we consider Pareto-undominated refinements of BNE in a Bayesian game with private information Γ. We first provide the definitions of Pareto-undominated and socially-maximal BNE.

Definition 6. *In a Bayesian game with private information Γ, a strategy profile g is Pareto dominated by a strategy profile g' if for all $t \in T$, $U_t(g') \geq U_t(g)$, with the strict inequality for at least one $t \in T$. Pareto-undominated BNE of Γ is a BNE that is not Pareto dominated by any other BNE.*

Definition 7. *In a Bayesian game with private information Γ, a strategy profile g is dominated in social welfare by a strategy profile g' if $\sum_{t\in T} U_t(g') > \sum_{t\in T} U_t(g)$. A socially-maximal BNE of Γ is a BNE that is not dominated in social welfare by any other BNE.*

As Pareto domination implies domination in social welfare in Γ, a socially-maximal BNE is clearly a Pareto-undominated BNE. Once there exists a socially-maximal BNE in a Bayesian game with private information, a Pareto-undominated BNE must exist. So in the remainder of this section, we only focus on socially-maximal BNE in Γ.

The next result gives the relation between socially-maximal equilibria in a Bayesian game with diffused and disparate information Γ and its reduced LGT $\mathcal{G}^\Gamma$.

Theorem 4. *Suppose that a Bayesian game with private information Γ satisfies Assumptions 2 and 3. If its induced LGT $\mathcal{G}^\Gamma$ has a socially-maximal Nash euqilibirum, then Γ has a socially-maximal BNE.*

The next result is thus a direct consequence of Proposition 1.

Corollary 2. *Under hypotheses of Corollary 1, there also exists a socially-maximal BNE.*

4 Conclusion

In this paper, Pareto-undominated and socially-maximal equilibria have been defined and examined in the setting of both LGTs and Bayesian games with diffused and disparate private information. The key technical method to deal with equilibria and their Pareto-undominated refinements in a Bayesian game is to reformulate the game into an LGT, prove the existence results in such an LGT, and bring the results back to the originally given Bayesian game. It is worth pointing out such a connection can be also established between an LGT and a Bayesian game with private and public information as considered in Fu et al. [11].[27]

Pareto-undominated and socially-maximal equilibria studied in this paper are immune to grand coalitional deviations to other equilibria. In finite-games, some other refinements of Nash equilibria, such as strong equilibria and coalition-proof equilibria, which are immune to individual or group deviations by other principals, have been well-established and studied in the literature.[28] Thus, a natural question to ask is whether one could also discuss coalition-proof and/or strong equilibria in general non-atomic games.[29] Another question of interest is under what general conditions one can provide a justification for reaching Pareto-undominated and/or socially-maximal equilibria in a game. This is to ask how to apply such equilibria as the equilibrium outcome in a broad spectrum of situations such as global games and many other scenarios that deal with a continuum of players, or diffused and disparate information. One possibility is that one can rely on randomized strategy equilibria, and see whether and how randomized equilibria can be reduced to undominated equilibria after uncertainty is resolved.[30] We hope to address both questions in subsequent work.

Appendix

We first present a result on distributions of correspondences which is culled from the corresponding results in Yu and Zhang [40] and Keisler and Sun [16].

Lemma 2. *Let X be a Polish space and $(\Omega, \mathcal{A}, P)$ a non-atomic probability space. In addition, if (i) X is a countable, or (ii) $(\Omega, \mathcal{A}, P)$ is a saturated probability space, the following results are valid:*

L1: Let $\{f_n\}$ be a sequence of measurable functions from Ω to X such that $\tau_n = P{f_n}^{-1}$ converges weakly to $\tau \in \mathcal{M}(X)$ as $n \to \infty$. Let $G(\omega) = cl\text{-}\mathrm{Lim}\,\{f_n(\omega)\}$.[31] *Then, $G(\omega)$ is nonempty for almost all ω, and there exists a measurable selection f of G such that $Pf^{-1} = \tau$.*

[27] As this point is rather technical, we skip the details. They are available from the authors on request.

[28] See Bernheim et al. [6] for discussions on the relationship of Pareto-undominated Nash equilibria, strong equilibria and coalition-proof equilibria in finite games.

[29] See Konishia et al. [23] for the discussion on strong Nash equilibrium in non-atomic games in the setting of Schmeidler [37], and in particular on their assumptions of "partial rivalry" and "independence of irrelevant choices."

[30] However, to model the non-cooperative aspects of randomized equilibria, one has to work with a process with a continuum of independent random variables, whereas subtle measurability issues arise since such a process itself usually are not measurable. See Sun [38] and its references for such issues and Fubini extension therein to solve them.

[31] For any sequence x_n in a topology space, denote cl-Lim $\{x_n\}$ the set of its limit points. For any sequence of sets A_n in a topology space, denote cl-Lim A_n the union of all such cl-Lim $\{x_n\}$ with $x_n \in A_n$ for all n.

L2: For any nonempty, compact-valued and measurable correspondence F from $(\Omega, \mathcal{A}, P)$ to X, $\mathscr{D}_F = \{Pf^{-1} : f$ is a measurable selection of $F\}$ is nonempty, compact and convex.

L3: Let F be a compact-valued correspondence from $(\Omega, \mathcal{A}, P)$ to X. Suppose that Y is a metric space and G is a closed-valued correspondence from $\Omega \times Y$ to X such that, for all $(\omega, y) \in \Omega \times Y$, $G(\omega, y) \subseteq F(\omega)$, for each fixed $y \in Y$, $G(\cdot, y)$ (denoted by G_y) is a measurable correspondence from $(\Omega, \mathcal{A}, P)$ to X, and for each fixed $\omega \in \Omega$, $G(\omega, \cdot)$ is upper hemicontinuous from Y to X. Then the correspondence $H(y) = \mathscr{D}_{G_y}$ is upper hemicontinuous from Y to $\mathcal{M}(X)$.

Proof of Theorem 1: Suppose that (i) or (ii) holds. We first show that $\mathscr{D}_D^\alpha$ is nonempty, compact and convex. Towards this end, let $\tilde{D}(i) = (\alpha(i), D(i))$ for all $i \in I$ and $\mathscr{D}_{\tilde{D}} = \{\lambda \tilde{g}^{-1} : \tilde{g}$ is a measurable selection of $\tilde{D}\}$. It is clear that $\mathscr{D}_D^\alpha = \mathscr{D}_{\tilde{D}}$. The correspondence D from I to $T \times A$ is nonempty, compact-valued and measurable, so is $\tilde{D}$. Thus, by Lemma 2(L2), we know that $\mathscr{D}_D^\alpha$ is compact, convex and nonempty.

Now for any $\tau \in \mathscr{D}_D^\alpha$ and $i \in I$, let $B(i, \tau) = \mathrm{argmax}_{a \in D(i)} v(i, a, \tau)$. We can appeal to the Berge's maximum theorem (see Aliprantis and Border [2, Theorem 17.31] for example), and the Kuratowski-Ryll-Nardzewski selection theorem to assert that $B(i, \cdot)$ is upper hemicontinuous and there exists a measurable selection from $B(\cdot, \tau)$. Let $\tilde{B}(i, \tau) = \{\alpha(i)\} \times B(i, \tau)$ for all $i \in I$ and $\tau \in \mathscr{D}_D^\alpha$. It is clear that $\tilde{B}(i, \cdot)$ is also upper hemicontinuous on D_D^α for each i and $\tilde{B}(i, \tau)$ is closed-valued for any given (i, τ). Now let $\Phi : \mathscr{D}_D^\alpha \longrightarrow \mathscr{D}_D^\alpha$ be a correspondence such that $\Phi(\tau) = \mathcal{D}_{\tilde{\mathcal{B}}(\cdot, \tau)}$. As there exists a measurable selection from the correspondence $B(\cdot, \tau)$, ϕ has nonempty values. Together with Lemma 2 (L2 and L3), it is easy to see that Φ is a nonempty, closed and convex valued, upper hemicontinuous correspondence from $\mathscr{D}_D^\alpha$ to $\mathscr{D}_D^\alpha$. Therefore, we can appeal to the Kakutani-Fan-Glicksberg fixed point theorem to guarantee the existence of Nash equilibria in $\mathcal{G}$. ■

We now provide proofs to Theorem 2 and Proposition 1 in Section 2. As the set of Nash equilibria in an LGT may not be compact, we have to work with induced societal responses directly to obtain the existence of Pareto-undomindated and socially-maximal Nash equilibria in an LGT.

Proof of Theorem 2: Suppose that $\mathcal{G}$ is such an LGT. Let F_D be the set of all strategy profiles of $\mathcal{G}$ and $\Sigma_D = \{\lambda(\alpha, g)^{-1} | g \in F_D\}$. It is easy to see that $\Sigma_D = \mathscr{D}_D^\alpha$. As shown in the proof of Theorem 1, Σ_D is nonempty and compact.

Let $\hat{F}_D$ be the set of all Nash equilibria of $\mathcal{G}$ and the set of all externalities under Nash equilibrium $\hat{\Sigma}_D = \{\lambda(\alpha, f)^{-1} | f \in \hat{F}_D\}$. We now show that $\hat{\Sigma}_D$ are nonempty and compact. Since (i) or (ii) holds, Theorem 1 ensures that $\hat{F}_D$ is nonempty. We only need to show the compactness of $\hat{\Sigma}_D$. Towards this end, let $\{\hat{\tau}_n\}$ be a sequence of $\hat{\Sigma}_D$ and $\{\hat{f}_n\}$ a sequence of $\hat{F}_D$ satisfying $\lambda(\alpha, \hat{f}_n)^{-1} = \hat{\tau}_n$, for all $n \in \mathbb{N}$. Because $\hat{\tau}_n$ is also a sequence of Σ_D, the compactness of Σ_D implies that there is a subsequence of $\{\hat{\tau}_n\}$ that converges weakly to a point $\hat{\tau} \in \Sigma_D$. Without loss of generality, let the subsequence be the sequence itself. Now define G to be $G(i) = \text{cl-Lim}\{(\alpha(i), \hat{f}_n(i))\}$, which is nonempty and included in $\{\alpha(i)\} \times D(i)$ since the whole sequence $\hat{f}_n(i)$ is from the compact set $D(i)$ for each i. As T and A are Polish, by Lemma 2(L1), there exists a measurable selection $(\alpha, \hat{f})$

of G such that $\hat{\tau} = \lambda(\alpha, \hat{f})^{-1}$. This is to say, $\{\lambda(\alpha, \hat{f}_n)^{-1}\}$ converges weakly to $\lambda(\alpha, \hat{f})^{-1}$ in Σ_D. Meanwhile, since $\{\hat{f}_n\} \in \hat{F}_D$, we know that for λ-almost all $i \in I$, for all $n \in \mathbb{N}$ and for any given $a \in D(i)$,

$$v(i, \hat{f}_n(i), \lambda(\alpha, \hat{f}_n)^{-1}) \geq v(i, a, \lambda(\alpha, \hat{f}_n)^{-1}).$$

Together with the continuity assumption of $v(i, \cdot, \cdot)$ on $D(i) \times \Sigma_D$ and the fact that $(\alpha, \hat{f})$ is a measurable selection of G, the above inequality implies that for λ-almost all $i \in I$,

$$v(i, \hat{f}(i), \lambda(\alpha, \hat{f})^{-1}) \geq v(i, a, \lambda(\alpha, \hat{f})^{-1}).$$

Since a is arbitrarily chosen from $D(i)$, then for λ-almost all $i \in I$, the above inequality holds for all $a \in D(i)$. This simply means that $\hat{f} \in \hat{F}_D$. Therefore, $\hat{\tau} \in \hat{\Sigma}_D$, and hence $\hat{\Sigma}_D$, the set of externalities under Nash equilibrium, is compact.

For each $i \in I$ and $\tau \in \Sigma_D$, let $m(i, \tau) = \max\{v(i, a, \tau) : a \in D(i)\}$. By Berge's maximum theorem, we know that $m(i, \cdot)$ is continuous on Σ_D, for each $i \in I$. Furthermore, the measurable maximum theorem (see Aliprantis and Border [2, Theorem 18.19] for example) implies that for every $\tau \in \Sigma_D$, $m(\cdot, \tau)$ is measurable. Construct a function $\bar{m} : \Sigma_D \to \mathbb{R}$ as follows,

$$\bar{m}(\tau) = \int_I h\left(m(i, \tau)\right) \mathrm{d}\lambda, \text{for all } \tau \in \Sigma_D,$$

where $h(\cdot)$ is a continuous strictly monotonic function (e.g., $\arctan(\cdot)$) mapping $\mathbb{R}$ to a bounded interval. We next show that $\bar{m}$ is continuous on Σ_D. Let $\{\tau_n\}$ be a sequence of Σ_D converging weakly to τ. By the continuity of $m(i, \cdot)$ and $h(\cdot)$, $\{h\left(m(i, \tau_n)\right)\}$ converges to $h\left(m(i, \tau)\right)$, for each $i \in I$. Furthermore, the dominated convergence theorem implies that the sequence $\{\bar{m}(\tau_n)\}$ converges to $\bar{m}(\tau)$. Hence, $\bar{m}$ is continuous on Σ_D. Therefore, there exists a $\tau^* \in \hat{\Sigma}_D$ such that $\bar{m}(\tau^*) \geq \bar{m}(\tau)$, for all $\tau \in \hat{\Sigma}_D$ since $\hat{\Sigma}_D$ is compact. This is to say, $f^* \in \hat{F}_D$ satisfying $\lambda(\alpha, f^*)^{-1} = \tau^*$ is a Pareto-undominated Nash equilibrium. This completes the proof. ■

Proof of Proposition 1: Throughout this proof, we shall follow the notation in the proof of Theorem 2. Assumption 1 implies that for any $\tau \in \Sigma_D$, $0 \leq |m(i, \tau)| \leq \phi(i)$ for λ-almost all $i \in I$. Since ϕ is integrable, $|m(\cdot, \tau)|$ is integrable and hence $m(\cdot, \tau)$ is also integrable. Similar to the continuity $\bar{m}$ on Σ_D in Theorem 1, one can show that the function $\tilde{m} : \Sigma_D \to \mathbb{R}$ defined by $\tilde{m}(\tau) = \int_I m(t, \mu) \mathrm{d}\lambda$ is also continuous. By the compactness of $\hat{\Sigma}_D$, there exists a $\tau^* \in \hat{\Sigma}_D$ such that $\tilde{m}(\tau^*) \geq \tilde{m}(\tau)$, for all $\tau \in \hat{\Sigma}_D$. Let f^* be a Nash equilibrium such that $\lambda(\alpha, f^*)^{-1} = \tau^*$. Suppose that f is an arbitrary Nash equilibrium in $\mathcal{G}$. Then $\tilde{m}(\tau^*) \geq \tilde{m}(\tau)$ for all $\tau \in \hat{\Sigma}_D$ implies that

$$\int_I v(i, f^*(i), \lambda(\alpha, f^*)^{-1}) \mathrm{d}\lambda \geq \int_I v(i, f(i), \lambda(\alpha, f)^{-1}) \mathrm{d}\lambda.$$

Therefore, f^* is a socially-optimal Nash equilibrium. ■

We shall follow the notation in Section 3.2 in the remaining proofs.

Proof of Lemma 1: Observe that $I^\Gamma \in \mathcal{I}^\Gamma$. For $E \in \mathcal{I}^\Gamma$, it can be written as $\bigcup_{m\in T} C_m \in \mathcal{I}^\Gamma$ where C_m is some element of $\mathcal{I}_m$ for all $m \in T$. The complement of E, $E^c = I \setminus E = \bigcup_{m\in T}(I_m \setminus C_m)$. Clearly, $E^c \in \mathcal{I}^\Gamma$. Now consider a sequence of sets $\{E_n\}_{n=1}^\infty$ in $\mathcal{I}^\Gamma$. By construction of $\mathcal{I}^\Gamma$, for any n, $E_n = \cup_{m\in T} C_m^n$ for some $C_m^n \in \mathcal{I}_m$ for all $m \in T$. Hence, $\cup_{n=1}^\infty E_n = \cup_{n=1}^\infty \cup_{m\in T} C_m^n = \cup_{m\in T}\cup_{n=1}^\infty C_m^n$. Moreover, as for all $m \in T$, $(I_m, \mathcal{I}_m, \lambda_m)$ is a probability space, it is clear that $\cup_{n=1}^\infty C_m^n \in \mathcal{I}_m$. This implies that $\mathcal{I}^\Gamma$ is also closed under countable union, and hence $\mathcal{I}^\Gamma$ is a σ-algebra. By the definition of λ^Γ, it is obvious that λ^Γ is a well-defined probability measure on $\mathcal{I}^\Gamma$. Furthermore, as $\mu\zeta_m^{-1}$ is non-atomic for all $m \in T$, λ_m is non-atomic, so is λ^Γ. Therefore, $(I^\Gamma, \mathcal{I}^\Gamma, \lambda^\Gamma)$ is a non-atomic probability space. ∎

Proof of Theorem 3: We first show that $\mathcal{G}^\Gamma$ is an LGT that is specified in Section 2.1 and satisfies Assumption 1. By Lemma 1, we know that the space of players of $\mathcal{G}^\Gamma$ is non-atomic. By the construction, we know that $D^\Gamma(i) = L_t(Proj_{Z_t}(i))$ if $i \in I_t$ for some $t \in T$, and L_t is compact-valued and measurable, it is clear that D^Γ is compact-valued and measurable. Furthermore, for any given $a_t \in A_t$ and $(\tau_m)_{m\in T} \in \prod_{m\in T} \mathscr{D}_{L_m}$, V_t is measurable from Z_t to $\mathbb{R}$. Since $\prod_{j\in I} \mathscr{D}_{L_m}$ is compact and $V_t(\cdot, z_t)$ in Section 3 is continuous on $\mathcal{A}$ for all $z_t \in Z_t$, it is obviously that for any fixed $z \in Z_t$, $\bar{V}_t$ is continuous from $A_t \times \prod_{m\in T} \mathscr{D}_{L_m}$ to $\mathbb{R}$; and for any fixed $(a_t, (\tau_m)_{m\in T})$, $\bar{V}_t$ is measurable on Z_t. Hence, for $i \in I_t$, $\bar{v}^\Gamma(i,\cdot,\cdot)$ is continuous on $D^\Gamma(i) \times \prod_{m\in T} \mathscr{D}_{L_m}$ and and $\bar{v}^\Gamma(\cdot,\cdot,(\tau_m)_{m\in T})$ is a measurable function on the graph of D^Γ for all $(\tau_m)_{m\in T} \in \prod_{m\in T} \mathscr{D}_{L_m}$. Furthermore, one can show that the function Φ which is used to construct v^Γ is a continuous mapping and a homeomorphism (see, for example, Khan et al. [20, Section 2]). Hence, $v^\Gamma(i,\cdot,\cdot)$ is continuous on $D^\Gamma(i) \times \mathscr{D}_D^\alpha$ for all $i \in I^\Gamma$, and $v^\Gamma(\cdot,\cdot,\tau)$ is a measurable function on the graph of D^Γ for all $\tau \in \mathscr{D}_{D^\Gamma}^{\alpha^\Gamma}$. Therefore, $\mathcal{G}^\Gamma$ is indeed an LGT specified in Section 2.1. In addition, because that $\bar{V}_t$ is uniformly integrable as pointed in Footnote 26 and $\bar{V}_t$ actually does not depend on τ_t, it is clear that Assumption 1 also holds for $\mathcal{G}^\Gamma$.

We now prove that Γ has a BNE if the induced LGT $\mathcal{G}^\Gamma$ has a Nash equilibrium. Suppose that f^* is a Nash equilibrium of $\mathcal{G}^\Gamma$. This is to say, for almost all $i \in I^\Gamma$,

$$v^\Gamma(i, f^*(i), \lambda^\Gamma(\alpha^\Gamma, f^*)^{-1}) \geq v^\Gamma(i, a, \lambda^\Gamma(\alpha^\Gamma, f^*)^{-1}) \text{ for all } a \in D^\Gamma(i).$$

For any $t \in T$, let $f_t^* : I_t \to A$ be defined by $f_t^*(i) = f^*(i)$ for all $i \in I_t$. Let $g^* = (g_m^*)_{m\in T}$ be such that for all $t \in T$, $g_t^* : Z_t \to A$ is a function that satisfies $g_t^*(z) = f_t^*(z,t)$ for all $z \in Z_t$. It is clear that g^* is a strategy profile of Γ. Furthermore, for all $t \in T$, it follows from (1) that $\lambda_t(B) = \frac{\lambda^\Gamma(B)}{\lambda^\Gamma(I_t)}$ for any $B \in \mathcal{I}_t$ and $\lambda^\Gamma(I_t) = \lambda^\Gamma(\alpha^\Gamma)^{-1}(t)$. Thus, $\phi(\lambda^\Gamma(\alpha^\Gamma, f^*)^{-1}) = \left(\lambda_m f_m^{*\,-1}\right)_{m\in T} = \left((\mu\zeta_m^{-1}) g_m^{*\,-1}\right)_{m\in T}$. By the construction of v^Γ, we have that for all $t \in T$ and for almost all $z \in Z_t$,

$$\bar{V}_t\left(z, g_t^*(z), ((\mu\zeta_m^{-1})g_m^{*\,-1})_{m\in T}\right) \geq \bar{V}_t\left(z, a, ((\mu\zeta_m^{-1})g_m^{*\,-1})_{m\in T}\right) \text{ for all } a \in L_t(z). \tag{2}$$

As Assumption 3 holds for Γ, by Fubini's Theorem, and the property of the regular conditional expectation, the payoff of player t in Γ under any strategy profile g can be written as follows:

$$\begin{aligned} U_t(g_t, g_{-t}) &= \int_{A_{-t}} \int_{Z_t} V_t\left(g_t(z_t), a_{-t}, z_t\right) \mathrm{d}\mu\zeta_t^{-1} \mathrm{d}\Pi_{m\neq t}(\mu\zeta_m^{-1}) g_m^{-1} \\ &= \int_{Z_t} \int_{A_{-t}} V_t(g_t(z_t), a_{-t}, z_t) \mathrm{d}\Pi_{m\neq t}(\mu\zeta_m^{-1}) g_m^{-1} \mathrm{d}\mu\zeta_t^{-1} \\ &= \int_{Z_t} \bar{V}^t(z_t, g_t(z_t), ((\mu\zeta_m^{-1}) g_m^{-1})_{m\in T}) \mathrm{d}\mu\zeta_t^{-1}, \end{aligned} \tag{3}$$

Therefore, by (2) and (3), we have that for all $t \in T$ and almost all $z_t \in Z_t$,

$$U_t(g_t^*, g_{-t}^*) \geq U_t(g_t', g_{-t}^*), \text{ for any pure strategy } g_t' \in \text{meas}(Z_t, L_t).$$

Hence, $g^* = (g_m^*)_{m\in T}$ is a BNE of Γ.

Finally, we prove that if Γ has a BNE then the induced LGT $\mathcal{G}^\Gamma$ has a Nash equilibrium. Suppose $g^* = (g_m^*)_{m\in T}$ is a BNE of Γ. For each $t \in T$, let $f_t^* : I_t \to A$ a function such that $f_t^*(z,t) = g_t^*(z)$ for all $z \in Z_t$ and define $f^* : I \to A$ by $f^*(i) = f_t^*(i)$ if $i \in I_t$. Apparently, f^* is measurable. We will show that f^* is a Nash equilibrium of $\mathcal{G}^\Gamma$.

Suppose not. Then there exists some $t \in T$ and a measurable subset $C_t \in I_t$ such that $\lambda^\Gamma(C_t) > 0$ and for almost all $i \in C_t$,

$$v^\Gamma(i, f^*(i), \lambda^\Gamma(\alpha^\Gamma, f^*)^{-1}) < v^\Gamma(i, a, \lambda^\Gamma(\alpha^\Gamma, f^*)^{-1}) \text{ for some } a \in D^\Gamma(i). \tag{4}$$

For all $i \in I^\Gamma$, let $B(i) = \arg\max_{a\in D^\Gamma(i)} v^\Gamma(i, a, \lambda^\Gamma(\alpha^\Gamma, f^*)^{-1}))$. By the measurable maximum theorem, B is nonempty compact-valued, and thus admits a measurable selection. Let $\varphi : I^\Gamma \to A$ be such a measurable selection of B. Let $g_t' : Z_t \to A$ be a function such that

$$g_t'(z_t) = \begin{cases} g_t^*(z_t) & \text{if } z_t \in Z_t \setminus E_t \\ \varphi(z_t, t) & \text{if } z_t \in E_t, \end{cases}$$

where $E_t \equiv \{z : (z,t) \in C_t\}$. Obviously, g_t' is also a pure strategy. By the construction of v^Γ through $\bar{V}_t$, (4) implies the following:

$$\bar{V}_t(z_t, g_t'(z_t), (\mu\zeta_m^{-1}) g_m^{*\,-1})_{m\in T}) \begin{cases} = \bar{V}_t(z_t, g_t^*(z_t), (\mu\zeta_m^{-1}) g_m^{*\,-1})_{m\in T}) & \text{if } z_t \in Z_t \setminus D_t \\ > \bar{V}_t(z_t, g_t^*(z_t), (\mu\zeta_m^{-1}) g_m^{*\,-1})_{m\in T}) & \text{if } z_t \in D_t. \end{cases}$$

Together with (3), we have $U(g_t', g_{-t}^*) > U_t(g_t^*, g_{-t}^*)$. But this contradicts the hypothesis that g is a BNE of Γ. Thus, f must be a Nash equilibrium of $\mathcal{G}^\Gamma$. The proof is now complete. ∎

Proof of Corollary 1: Let Γ be a Bayesian game with private information that satisfies Assumptions 2 and 3, and $\mathcal{G}^\Gamma$ be its induced LGT. If (i) holds, as A_t is countable for all $t \in T$ and T is finite, then condition (i) in Theorem 1 holds for $\mathcal{G}^\Gamma$. If (ii) holds, it is clear that the player space

$(I^\Gamma, \mathcal{I}^\Gamma, \lambda^\Gamma)$ of $\mathcal{G}^\Gamma$ is saturated, then condition (ii) in Theorem 1 holds for $\mathcal{G}^\Gamma$. In either case, there exists a Nash equilibrium of $\mathcal{G}^\Gamma$. Therefore, we can apply Theorem 3 to assert that there exists a BNE in Γ. ∎

Proof of Theorem 4: Let Γ be a Bayesian game that satisfies all assumptions. In its induced LGT $\mathcal{G}^\Gamma$, let $f^s : I^\Gamma \to A$ be a socially-maximal Nash equilibrium. Let $g^s = (g^s_m)_{m\in T}$ be such that for all $t \in T$, $g^s_t : Z_t \to A$ is a function that satisfies $g^s_t(z) = f^s_t(z,t)$ for all $z \in Z_t$ where $f^s_t : I_t \to A$ is defined by $f^s_t(i) = f^s(i)$ for all $i \in I_t$. As shown in the proof of Theorem 3, g^s is a BNE of Γ. We now show that it is also a socially-maximal BNE of Γ.

Suppose not. Then there exists some other BNE g^* of Γ such that

$$\sum_{t\in T} U_t(g^*) > \sum_{t\in T} U_t(g^s). \tag{5}$$

For each $t \in T$, let $f^*_t : I_t \to A$ by $f^*_t(z,t) = g^*_t(z)$ for all $z \in Z_t$ and define $f^* : I^\Gamma \to A$ by $f^*(i) = f^*_t(i)$ if $i \in I_t$. Clearly, f^* is a Nash equilibrium in $\mathcal{G}^\Gamma$ by the proof of Theorem 3. Since f^s is a socially-maximal Nash equilibrium in $\mathcal{G}^\Gamma$, we have

$$\int_{i\in I^\Gamma} v^\Gamma(i, f^s(i), \lambda^\Gamma(\alpha^\Gamma, f^s)^{-1}) \mathrm{d}\lambda^\Gamma \geq \int_{i\in I^\Gamma} v^\Gamma(i, f^*(i), \lambda^\Gamma(\alpha^\Gamma, f^*)^{-1}) \mathrm{d}\lambda^\Gamma.$$

From the proof of Theorem 3, we know that for all $t \in T$, $\phi(\lambda^\Gamma(\alpha^\Gamma, f^*)^{-1}) = \big(\lambda_m f^{*\,-1}_m\big)_{m\in T} = \big((\mu\zeta_m^{-1}) g^{*\,-1}_m\big)_{m\in T}$. It is also easy to see that $\phi^{-1}\big(\big((\mu\zeta_m^{-1}) g^{s\,-1}_m\big)_{m\in T}\big) = \phi^{-1}\big(\big(\lambda_m f^{s\,-1}_m\big)_{m\in T}\big) = \lambda^\Gamma(\alpha^\Gamma, f^s)^{-1}$. Meanwhile, for all $t \in T$, it follows from (1) that $\lambda_t(B) = \frac{\lambda^\Gamma(B)}{\lambda^\Gamma(I_t)} = |T|\lambda^\Gamma(B)$ for any $B \in \mathcal{I}_t$. Thus,

$$\begin{aligned}
\sum_{i\in I} U_t(g^s) &= \sum_{t\in T} \int_{Z_t} \bar{V}_t(z_t, g^s_t(z_t), (\mu\zeta_m^{-1}) g^{s\,-1}_m)_{m\in T}) \mathrm{d}\mu\zeta_t^{-1} \\
&= \sum_{t\in T} \int_{I_t} \bar{v}^\Gamma(i, f^s_t(i), \big(\lambda_m f^{s\,-1}_m\big)_{m\in T}) \mathrm{d}\lambda_t \\
&= |T| \sum_{t\in T} \int_{I_t} \bar{v}^\Gamma(i, f^s(i), \big(\lambda_m f^{s\,-1}_m\big)_{m\in T}) \mathrm{d}\lambda^\Gamma \\
&= |T| \int_{I^\Gamma} \bar{v}^\Gamma(i, f^s(i), \big(\lambda_m f^{s\,-1}_m\big)_{m\in T}) \mathrm{d}\lambda^\Gamma \\
&= |T| \int_{I^\Gamma} v^\Gamma(i, f^s(i), \lambda^\Gamma(\alpha^\Gamma, f^s)^{-1}) \mathrm{d}\lambda \\
&\geq |T| \int_{I^\Gamma} v^\Gamma(i, f^*(i), \lambda^\Gamma(\alpha^\Gamma, f^*)^{-1}) \mathrm{d}\lambda = \ldots = \sum_{t\in T} U_t(g^*),
\end{aligned}$$

which contradicts (5). Therefore, there must exist a socially-maximal BNE of Γ if there exists a socially-maximal Nash equilibrium in its induced LGT $\mathcal{G}^\Gamma$. ∎

Proofs of Corollaries 2: By Theorem 3, Assumption 1 holds for the induced LGT $\mathcal{G}^{\Gamma}$. Similar to the proof of Corollary 1, one can show that all assumptions in Proposition 1 hold for $\mathcal{G}^{\Gamma}$, and therefore there exists a socially-maximal Nash equilibrium in $\mathcal{G}^{\Gamma}$. One can now appeal to Theorem 4 to assert the existence of a socially-maximal BNE in Γ. ∎

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